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Asymptotic Rejection of General Periodic Disturbances in Output-feedback Nonlinear Systems

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Abstract

This paper deals with asymptotic rejection of general periodic disturbances in nonlinear systems. A novel approach based on integrations over the half of disturbance period is proposed to estimate disturbances which may not be harmonic or even smooth. The proposed algorithm uses the period and the knowledge of the basic disturbance patterns, while the amplitude and phase of the disturbances are unknown, and they are estimated. Based on the estimated disturbance, the disturbance rejection algorithm is then proposed for linear systems and nonlinear systems in the output feedback format, which asymptotically rejects the periodical disturbances. Proposed algorithm extends asymptotic rejection of unknown disturbances from sinusoidal disturbances to a more general class of periodical disturbances which includes zero-mean square waves, symmetric triangular waves etc.

I. INTRODUCTION

Recently, tremendous progresses have been reported in rejecting sinusoidal disturbances. Internal model principle plays an important role in the asymptotic rejection of sinusoidal disturbances. When the disturbance frequency is known, an internal model can be designed to generate the desired feed-forward input so that the disturbance can be rejected. A result for asymptotically rejecting sinusoidal disturbances is shown [1] for uncertain nonlinear systems in the output feedback form. Even for the case of sinusoidal disturbances of unknown frequencies, a series of results have been published for rejecting disturbances [2], [3], [4], [5]. A related problem is formulated as output regulation of nonlinear systems with the aim of rejecting disturbances and tracking desired trajectories. The local results of output regulation with known exosystems are shown in [6], [7], [8] and a global result for nonlinear systems in the output feedback is shown in [9]. Again, the internal model principle is the key to the success in output regulation. It is easy to model the sinusoidal disturbances as the output of linear exosystems, and the internal model can then be designed accordingly. This is not the case for other general periodical disturbances such as square wave disturbances. Until now there is not any report on asymptotically rejecting of square wave disturbances. If the period is long enough, integral actions in the system may reject the disturbance, but it is not in the sense of asymptotic rejection, ie, after the disturbance changes value, it will take some time to settle down again. Therefore the integral action is not good enough to reject square-wave disturbances.

Most of the internal model based methods make use of the information of linear exosystem with the knowledge of disturbance period to reject disturbances such as sinusoidal disturbances. This is not the case for general periodic disturbances. If we consider a square wave as $a\text{sign}(\sin(\omega t))$, the internal model is nonlinear due to the sign function, and the internal model method is difficult to apply. However, considering the sinusoidal disturbance with known frequency, the internal model method in fact provides the estimation of disturbance amplitude and phase. This motivates the design for disturbance rejection of general periodical disturbances in the direction of estimating the unknown disturbance from the information of the period and the wave pattern. If an estimate of the unknown disturbance is obtained, control design for disturbance rejection can then be followed.

In this paper, estimation of general periodical disturbance is addressed by exploiting the properties of general periodical signals, especially the properties after the integration over half of the period. A novel half-period integral operator and the delay operator are introduced to exploit the properties of disturbances. A series of results are obtained for the two operators on the general period signals, based on which estimation algorithms are proposed. With the estimated disturbance, control design is then proposed for disturbance rejection with stability. The nice property of the estimate ensures the asymptotic rejection of general periodical disturbances under the proposed control for nonlinear systems in the output feedback form. A simpler control algorithm is proposed for linear systems. An example is included to demonstrate the proposed estimation and control algorithm in different disturbances such as square wave, triangular wave and sinusoidal disturbances.

II. Problem formulation

Consider a single-input-single-output nonlinear system which can be transformed into the output feedback form

$$\begin{aligned} \dot{x} &= A_c x + \psi(y) + b(u - w) \\ y &= Cx \end{aligned} \tag{1}$$

with

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T, b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_\rho \\ \vdots \\ b_n \end{bmatrix}$$

where $x \in R^n$ is the state vector, $u \in R$ is the control, ψ, is a known nonlinear smooth vector field in R^n with $\psi(0) = 0$, $w \in R$ is a periodical disturbance.

Assumption 1. The disturbance can be expressed as

$$w(t) = aw_b(t + \phi) \tag{2}$$

where the unknown constants a and ϕ are referred to as amplitude and phase, and $w_b(t)$ is a known function satisfying the following

A1.1 $w_b(t+T) = w_b(t)$ with T, the known period.
A1.2 $w_b(-t) = -w_b(t)$.
A1.3 $w_b(\frac{T}{4} - t) = w_b(\frac{T}{4} + t)$.
A1.4 There exists a δ, $0 < \delta < \frac{T}{4}$, such that for $t \in (0, \delta)$, $w_b(t) < K_b t^l$ with K_b and l are positive reals, and $w_b(t) \geq K_b \delta^l$ for $t \in [\delta, \frac{T}{4}]$.
Assumption 2. For any function $\delta(t)$, the following inequality holds for the basic wave form function $w_b(t)$:

$$\int_0^T |w_b(t) - w_b(t+\delta(t))| dt \leq K_\delta \int_0^T |\delta(t)| dt \tag{3}$$

where K_δ is positive real constant.
Remark 1: Assumption 1 specifies the types of disturbances considered in this paper. It can be seen that sinusoidal functions, square waves and triangular waves with zero means all satisfy. Assumption 2 is needed to establish the convergence of disturbance estimation for general waveforms, and all the wave forms with bounded first order derivatives will ensure that Assumption 2 is satisfied. Similar but different conditions may be stated for non-differentiable wave forms such as square waves and triangular waves. They are used in the proof for estimation results for square wave and triangular waves.

The problem considered in this paper is to design a dynamic feedback control law u so that the overall system is stable and the unknown disturbance $w(t)$ is asymptotically rejected in the sense that $\lim_{t\to\infty} y(t) = 0$. The asymptotic rejection algorithm proposed here adopts an indirect approach, ie, the disturbance is estimated and then the estimated disturbance is used for control design for disturbance rejection. For the control design of nonlinear systems, the following assumption is also needed.
Assumption 3. The system is minimum phase, ie, the zeros of polynomial $\mathbf{B}(s) = \sum_{i=\rho}^{n} b_i s^{n-i}$ have negative real parts.
Remark 2: We only consider the matched disturbances or input disturbances in (1). Many unmatched disturbances and even some cases of output regulation can be converted to the form as shown in (1) with $-w$ being interpreted as the desired feed-forward input. Therefore the problem considered in this paper can also interpreted as the problems for general disturbance rejection and output regulation of which the desired input satisfies the conditions specified in Assumptions 1 and 2, with the output measurement that does not contain the disturbance.

III. Properties of half-period integration and delay operators

Since the basic disturbance pattern is described by the function $w_b(t)$, the disturbance can be reproduced if the amplitude a and phase ϕ can be estimated. In this section, the periodic property and wave pattern properties described in Assumption 1 will be exploited to design estimation algorithms for a and ϕ.

Define the half-period integration operator $\mathcal{I}$ and the quarter-period delay operator $\mathcal{D}$ as

$$\begin{aligned} \mathcal{I} \circ f(t) &:= \mathcal{I}(f(t)) = \int_{t-\frac{T}{2}}^{t} f(s) ds \\ \mathcal{D} \circ f(t) &:= \mathcal{D}(f(t)) = f(t - \frac{T}{4}) \end{aligned} \tag{4}$$

For the convenience of notations, we often write $\mathcal{I} \circ f$ and $\mathcal{D} \circ f$ as $\mathcal{I} f$ and $\mathcal{D} f$ when no confusions are caused. It is easy to see the following properties of the introduced operators such as

$$\mathcal{I}\frac{df(t)}{dt} = f(t) - f(t - \frac{T}{2}) \tag{5}$$

for a C^1 function f, and

$$\mathcal{D}^k w(t) = \mathcal{D}^{\bar{k}} w(t) \tag{6}$$

with $\bar{k} = \mod (k, 4)$, for a periodic function $w(t)$ with period T. The operations of $\mathcal{D}$ and $\mathcal{I}$ can be swapped in sequence, ie, $\mathcal{D} \circ \mathcal{I} \circ f = \mathcal{I} \circ \mathcal{D} \circ f$. An important property is described in the following lemma.

Lemma 3.1 If a function $f(t)$ satisfies the conditions specified in Assumption 1, so does the function $g(t)$ defined by $g = \mathcal{D}^3 \circ \mathcal{I} \circ f$.

Proof: The conditions A1.1 to A1.3 of $g(t)$ can be established by directly evaluating the integrations introduced by the operations $\mathcal{D}^3 \circ \mathcal{I}$. For A1.4, it is easy to obtain that

$$g(t) \;=\; \int_{t-\frac{T}{4}}^{t+\frac{T}{4}} f(s)ds \tag{7}$$

For $0 < t < \frac{T}{4}$, it can be obtained that

$$g(t) \;=\; \int_{t-\frac{T}{4}}^{\frac{T}{4}-t} f(s)ds + \int_{\frac{T}{4}-t}^{\frac{T}{4}} f(s)ds + \int_{\frac{T}{4}}^{\frac{T}{4}+t} f(s)ds = 2\int_{\frac{T}{4}-t}^{\frac{T}{4}} f(s)ds \tag{8}$$

Let $\delta' = \frac{T}{4} - \delta$. Consider $t \in (0, \delta')$. The corresponding range for s in (8) is between $\frac{T}{4}$ and δ, and therefore it can be obtained that

$$g(t) \;\geq\; 2K_b\delta^l t := K_b' t^{l'} \tag{9}$$

with $K' = 2K_b\delta^l$ and $l' = 1$. Consider the case for $t \in [\delta', \frac{T}{4})$. It follows from (8) and the fact $f(t) \geq 0$ for $t \in (0, \frac{T}{4})$ that

$$g(t) \geq 2\int_{\frac{T}{4}-\delta'}^{\frac{T}{4}} f(s)ds \geq 2\delta K_b\delta^l = K'\delta'^{l'} \tag{10}$$

Hence the proof is completed.

Consider a disturbance passing through a linear dynamic system described by the following differential equation

$$\frac{d^m y}{dt^m} + \beta_1\frac{d^{m-1}y}{dt^{m-1}} + \ldots + \beta_m y = w(t) \tag{11}$$

where β_i, for $i = 1, \ldots, m$ are constants, and y is the measurable output. If the disturbance $w(t)$ satisfies Assumption 1, then the phase and gain can be calculated as shown in the following lemma.

Lemma 3.2 If y is the output in (11) and the input $w(t)$ satisfies Assumption 1, then the phase and gain can be calculated directly from $y(t)$ by

$$a \;=\; \frac{\mathcal{I} \circ |\bar{y}(t)|}{\mathcal{I} \circ |w_{b,m}(t)|} \tag{12}$$

$$\phi \;=\; \phi_1 - \phi_2 \tag{13}$$

where

$$\bar{y}(t) = \mathcal{D}^{\bar{m}} \sum_{i}^{m} \beta_i \mathcal{I}^i (1-\mathcal{D}^2)^{m-i} y(t) \tag{14}$$

$$w_{b,m}(t) = \mathcal{D}^{\bar{m}} \mathcal{I}^m w_b(t) \tag{15}$$

$$\phi_1(t) = \frac{1}{2}(\mathcal{I} \circ \text{sign}(\bar{y}(t)) + \frac{T}{2})\text{sign}(\bar{y}(t)) \tag{16}$$

$$\phi_2(t) = \frac{1}{2}(\mathcal{I} \circ \text{sign}(w_b(t)) + \frac{T}{2})\text{sign}(w_b(t)) \tag{17}$$

with $\beta_0 = 1$ and $\bar{m} = \text{mod}(3m, 4)$

Proof: Observing that $\mathcal{I} \circ \frac{dy}{dt} = (1-\mathcal{D}^2) \circ y(t)$, it can be shown that

$$\bar{y}(t) = \mathcal{D}^{\bar{m}} \mathcal{I}^m w_b(t) \tag{18}$$

On the other hand, if we define

$$w_{b,i}(t) = \mathcal{I} w_{b,i-1}(t), \text{ for } i = 1, \ldots, m \tag{19}$$

with $w_{b,0}(t) := w_b(t)$. It can be shown that

$$\mathcal{D}^3 \mathcal{I} w(t) = a \mathcal{D}^3 \mathcal{I} w_b(t+\phi) = a w_{b,1}(t+\phi) \tag{20}$$

and similarly

$$\bar{y}(t) = a w_{b,m}(t+\phi) \tag{21}$$

Then the result shown in (12) follows by integrating the absolute value of the above equation over half a period. Consider for the primary period where the operant in $[0, T)$,

$$\text{sign}(w_{b,m}(t+\phi)) = \begin{cases} 1 & \text{if } -\phi < t < \frac{T}{2} - \phi \\ -1 & \text{if } \frac{T}{2} - \phi < t < T - \phi \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

and

$$\mathcal{I} \circ \text{sign}(w_{b,m}(t+\phi)) = \begin{cases} 2(t+\phi) - \frac{T}{2} & \text{if } -\phi < t < \frac{T}{2} - \phi \\ \frac{3}{2}T - 2(t+\phi) & \text{if } \frac{T}{2} - \phi < t < T - \phi \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

and therefore

$$\phi_1 = \begin{cases} t+\phi & \text{if } 0 < t < \frac{T}{2} - \phi \\ t+\phi-T & \text{if } \frac{T}{2} - \phi < t < T \end{cases} \tag{24}$$

Similarly, it can be shown that

$$\phi_2 = \begin{cases} t & \text{if } 0 < t < \frac{T}{2} \\ t-T & \text{if } \frac{T}{2} < t < T \end{cases} \tag{25}$$

It can then obtained that

$$\phi_1 - \phi_2 = \begin{cases} \phi & \text{if } 0 \leq t < \frac{T}{2} - \phi \\ \phi - T & \text{if } \frac{T}{2} - \phi \leq t < \frac{T}{2} - \phi \\ \phi & \text{if } \frac{T}{2} < t < T \end{cases} \tag{26}$$

Note that for the phase calculation, $\phi - T$ is equivalent to ϕ, and hence, the proof is completed.

IV. Disturbance estimation

In order to extract the contribution in system state due to the disturbances, the following filter is designed:

$$\dot{p} \;=\; (A_c + kC)p + \phi(y) + bu - ky \tag{27}$$

where $p \in R^n$, $k \in R^n$ is chosen so that

$$\begin{aligned} K(s) \;&:=\; s^n - \sum_{i=1}^{n} k_i s^{n-i} \\ &=\; B(s)(s^\rho + \lambda_1 s^{\rho-1} + \ldots + \lambda_\rho)/b_\rho \end{aligned} \tag{28}$$

with λ_i being positive real design parameters such that $(s^\rho + \lambda_1 s^{\rho-1} + \ldots + \lambda_\rho)$ is Hurwitz. An estimate of w is given by

$$\hat{w}(t) = \hat{a} w_b(\hat{\phi}_1) \tag{29}$$

where

$$\hat{a} \;=\; \frac{\mathcal{I} \circ |\bar{w}(t)|}{\mathcal{I} \circ |w_{b,\rho}(t)|} \tag{30}$$

$$\hat{\phi}_1(t) \;=\; \frac{1}{2}(\mathcal{I} \circ \text{sign}(\bar{w}(t)) + \frac{T}{2})\text{sign}(\bar{w}(t)) \tag{31}$$

with

$$\bar{w}(t) \;=\; \mathcal{Q} \circ (p_1 - y) \tag{32}$$

$$\mathcal{Q} \;=\; \mathcal{D}^{\bar{\rho}} \sum_{i}^{\rho} \lambda_i \mathcal{I}^i (1 - \mathcal{D}^2)^{\rho - i} \tag{33}$$

and $\bar{\rho} = \text{mod}(3\rho, 4)$

Theorem 4.1 If the disturbance in (1) satisfies the conditions specified in Assumptions 1 and 2, then the estimate given in (29) converges to the actual disturbance in L_p, ie., $w - \hat{w} \in L_p$ for $p = 1, 2$ and ∞.

Proof: It is easy to see that both w and $\hat{w}$ are bounded signals, and therefore $w - \hat{w} \in L_\infty$. To complete the proof, it only needs to show the case for L_1, as $w - \hat{w} \in L_1 \cap L_\infty$ implies $w - \hat{w} \in L_2$

Consider a dummy filter

$$\dot{q} \;=\; (A_c + kC)q + bw \tag{34}$$

where q denotes the steady state only. Let $e = x - (p - q)$, then it is easy to show that

$$\dot{e} = (A_c + kC)e \tag{35}$$

Since $(A_c + kC)$ is Hurwitz, it can be shown that

$$\|e(t)\| \leq K_e e^{-\lambda_e t} \tag{36}$$

for some positive real constants K_e and λ_e. From the special structure of k chosen in (28), it can be shown that, assuming that the initial states associated with the zero dynamics are zero,

$$\frac{d^\rho q_1}{dt^\rho} + \lambda_1 \frac{d^{\rho-1} q_1}{dt^{\rho-1}} + \ldots + \lambda_\rho q_1 = w(t) \tag{37}$$

Then Lemma 3.2 can be used to calculate the amplitude and phase from the q_1, if q_1 would be available. Therefore if we define

$$\begin{aligned} \bar{q}_1(t) &= \mathcal{Q} \circ q_1(t) && (38)\\ \phi_1(t) &= \frac{1}{2}(\mathcal{I} \circ \text{sign}(\bar{q}(t)) + \frac{T}{2})\text{sign}(\bar{q}(t)) && (39)\\ & && (40) \end{aligned}$$

then we have $\phi_1 = t + \phi$ or $\phi_1 = t + \phi - T$ as shown in the proof of Lemma 3.2 and

$$\begin{aligned} a &= \frac{\mathcal{I} \circ |\bar{q}_1(t)|}{\mathcal{I} \circ |w_{b,\rho}(t)|} && (41)\\ w(t) &= a w_b(\phi_1) && (42) \end{aligned}$$

Let $\tilde{w} = w - \hat{w}$, then it can be expressed as

$$\begin{aligned} \tilde{w} &= a w_b(\phi_1) - \hat{a} w_b(\hat{\phi}_1) && (43)\\ &= \tilde{a} w_b(\phi_1) + \hat{a}(w_b(\phi_1) - w_b(\hat{\phi}_1)) && (44) \end{aligned}$$

where $\tilde{a} = a - \hat{a}$. The proof can be completed by establishing the boundedness of $\hat{a}$ and $(w_b(\phi_1) - w_b(\hat{\phi}_1))$, $\tilde{a} \in L_p$ for $p = 1, 2$ and ∞. The boundedness of $\hat{a}$ follows directly from (30) with the boundedness of $\bar{w}(t)$ which is in turn implied by the boundedness of $p_1 - y = q_1 - e_1$.

In order to show $\tilde{a} \in L_p$, the property of $e_w := \bar{q}_1 - \bar{w}$ needs to be investigated. From the definitions, it can be obtained that

$$e_w = \mathcal{Q} \circ q_1 - \mathcal{Q} \circ (p_1 - y) \tag{45}$$

Since $\mathcal{Q}$ is just a combination of linear operators, we have

$$e_w = \mathcal{Q} \circ (q_1 - (p_1 - y)) = \mathcal{Q} \circ e_1 \tag{46}$$

From (36), and the above equation, it can be obtained that

$$e_w(t) \leq K_w e^{-\lambda_e t} \tag{47}$$

for a positive constant K_w. From (30) and (41) it can be obtained that

$$|\tilde{a}| = \frac{|\mathcal{I} \circ (|\bar{q}_1(t)| - |\bar{w}(t)|)|}{\mathcal{I} \circ |w_{b,\rho}(t)|} \leq \frac{\mathcal{I} \circ |e_w|}{\mathcal{I} \circ |w_{b,\rho}(t)|} \tag{48}$$

Then from (47), it can be obtained that $\tilde{a} \in L_p$.

The next step is to establish $e_q := \text{sign}(\bar{q}(t)) - \text{sign}(\bar{w}(t)) \in L_p$. Since e_q is bounded, we only need to establish $e_q \in L_1$, which is equivalent to show that

$$J_\infty = \int_0^\infty |\text{sign}(\bar{q}_1) - \text{sign}(\bar{w}(t))| dt < \infty \tag{49}$$

Define

$$J_i = \int_{(i-1)T}^{iT} |\text{sign}(\bar{q}_1(t)) - \text{sign}(\bar{q}_1(t) - e_w(t))|dt \tag{50}$$

It follows that

$$J_\infty = \sum_{i=0}^{\infty} J_i \tag{51}$$

Since $|e_w(t)|$ is bounded by an exponentially decaying function, there exists an $\bar{i}$ such that for $t > \bar{i}T$, $|e_w(t)| < K_b\delta^l$. Therefore, for $i > \bar{i}$, it can be shown that

$$J_i < 4(\frac{K_w}{K_b})^{1/l} e^{-\lambda_e iT/l} \tag{52}$$

Hence J_∞ exists and $e_q \in L_p$ for $p = 1, 2$ and ∞.

With the result $e_q \in L_p$ for $p = 1, 2$ and ∞, it can be established $e_\phi := \phi_1 - \hat{\phi}_1 \in L_p$ from (17) and (31). Subsequently, applying the property specified in Assumption 2, it can be shown that $w_b(\phi_1) - w_b(\hat{\phi}_1) = w_b(\phi_1) - w_b(\phi_1 - e_\phi) \in L_p$, and this completes the proof.

For the disturbances with straight lines in the wave form, the implementation of the estimation algorithm may be simplified. In the following, the simplified results are given for square waves and triangular waves.

Corollary 4.2 If the basic wave form for the disturbance in (1) is the square wave form described by

$$w_b(t) = \begin{cases} 1 & \text{if } 0 < t < \frac{T}{2} \\ -1 & \text{if } \frac{T}{2} < t < T \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

the disturbance $w(t) = aw_b(t + \phi)$ can be estimated by

$$\hat{w}(t) = \hat{a}\text{sign}(\bar{w}(t)) \tag{54}$$

with the property that $w - \hat{w} \in L_p$ for $p = 1, 2$ and ∞

Corollary 4.3 If the basic wave form for the disturbance in (1) is the triangular form described by

$$w_b(t) = \begin{cases} \frac{4}{T}t & \text{if } 0 \le t < \frac{T}{4} \\ 2 - \frac{4}{T}t & \text{if } \frac{T}{4} \le t < \frac{3}{4}T \\ -4 + \frac{4}{T}t & \text{if } \frac{3}{4}T \le t < T \end{cases} \tag{55}$$

the disturbance $w(t) = aw_b(t + \phi)$ can be estimated by

$$\hat{w}(t) = \frac{2}{T}\hat{a}\mathcal{D}^3 I \circ \text{sign}(\bar{w}(t)) \tag{56}$$

Remark 3: Assumption 2 is not used for the results shown in Corollaries 4.2 and 4.3. Their special wave forms lead to a similar result as shown in (52), and Corollaries 4.2 and 4.3 can be proved in a similar way.

V. Disturbance rejection with stabilization

A control algorithm is to be presented for disturbance rejection using a disturbance estimate $\hat{w}$ which satisfies $\tilde{w} \in L_p$ with $p = 1, 2, \infty$. A state observer is designed as

$$\dot{\hat{x}} \;=\; (A_c + kC)\hat{x} + \psi(y) + b(u - \hat{w}) - ky \tag{57}$$

Let $\tilde{x} := x - \hat{x}$, It follows that

$$\dot{\tilde{x}} := (A_c + kC)\tilde{x} - b\tilde{w} \tag{58}$$

Control design can then be carried out using backstepping based on (57). For the backstepping design, the following notations are used:

$$z_1 \;=\; y = x_1 \tag{59}$$

$$z_i \;=\; \hat{x}_i - \alpha_{i-1}, i = 2, \dots \rho \tag{60}$$

where α_i are the stabilizing functions designed in the backstepping procedures. The design starts from the dynamics of z_1 given by

$$\dot{z}_1 = x_2 + \psi_1(y) = z_2 + \alpha_1 + \psi_1(y) + \tilde{x}_1 \tag{61}$$

The first stabilizing function α_1 is designed as

$$\alpha_1 = -c_1 z_1 - d_1 z_1 - \psi_1(y) \tag{62}$$

where c_i and d_i are the positive real design parameters, for $i = 1, \dots, \rho$. The subsequent stabilizing functions are designed as

$$\begin{aligned}\alpha_i \;=\;& z_{i-1} - c_i z_i - d_i(\frac{\partial \alpha_{i-1}}{\partial y})^2 z_i - k_i \hat{x}_1 + k_i y - \psi_i(y) + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_j}\dot{\hat{x}}_j \\ & + \frac{\partial \alpha_{i-1}}{\partial y}(\hat{x}_2 + \psi_1(y))\end{aligned} \tag{63}$$

for $\rho = 2, \dots, \rho$. Finally the control input is given by

$$u = \hat{w} + \frac{\alpha_\rho - \hat{x}_{\rho+1}}{b_\rho} \tag{64}$$

The proposed control ensures the asymptotic rejection of the disturbance and the boundedness of all the variables in the closed-loop system. The stability result is summarized in the following theorem.

Theorem 5.1 For a system (1) satisfying Assumptions 1, 2 and 3, the control input u given in (64) with the estimated disturbance $\hat{w}$ ensures the asymptotic rejection of the unknown disturbance, ie., $\lim_{t\to\infty} y(t) = 0$, and the boundedness of the other variables in the system.

Proof: Define

$$V_x = \tilde{x}^T P \tilde{x} \tag{65}$$

where P is a positive definite matrix satisfying

$$P(A_o + kC) + (A_o + kC)^T P = -3I \tag{66}$$

From (58), it can be obtained

$$\dot{V}_x = -3\tilde{x}^T\tilde{x} - 2\tilde{x}^T Pb\tilde{w} \leq -2\tilde{x}^T\tilde{x} + \|Pb\|^2\tilde{w}^2 \quad (67)$$

Define $V_z = \frac{1}{2}\sum_{i=1}^{\rho} z_i^2$ and it can be shown that

$$\begin{aligned} \dot{V}_z &= \sum_{i=1}^{\rho}(-c_i z_i^2 - d_i(\frac{\partial\alpha_{i-1}}{\partial y})^2 z_i^2 - \frac{\partial\alpha_{i-1}}{\partial y} z_i\tilde{x}_2) \\ &\leq \sum_{i=1}^{\rho}(-c_i z_i^2) + \gamma\tilde{x}_2^2 \end{aligned} \quad (68)$$

where $\gamma = \sum_{i=1}^{\rho}\frac{1}{4d_i}$ and $\frac{\partial\alpha_{-1}}{\partial y} := -1$. Let

$$V = V_z + \gamma V_x \quad (69)$$

From (67) and (68), it can be obtained that

$$\begin{aligned} \dot{V} &\leq -\sum_{i=1}^{\rho} c_i z_i^2 - \gamma\tilde{x}^T\tilde{x} + \gamma\|Pb\|^2\tilde{w}^2 \\ &\leq -\lambda V + \gamma\|Pb\|^2\tilde{w}^2 \end{aligned} \quad (70)$$

where $\lambda = \min\{2\min_{i=1,\ldots,\rho} c_i, \frac{\gamma}{\lambda_{max}(P)}\}$ with $\lambda_{max}(P)$ being the maximum eigenvalue of P. it can be concluded, using the comparison lemma [10], that

$$V(t) \leq \bar{V}(t) \quad (71)$$

where $\bar{V}(t)$ is generated by

$$\dot{\bar{V}} = -\lambda\bar{V} + \gamma\|Pb\|^2\tilde{w}^2, \bar{V}(0) = V(0) \quad (72)$$

With $\tilde{w}^2 \in L_1 \cap L_\infty$, as from Lemma 3.1, it can be concluded that $\bar{V} \in L_1 \cap L_\infty$ from the input-output stability theory [10], and hence $V \in L_1 \cap L_\infty$. The boundedness of V implies the boundedness of $\tilde{x}$ and z_i for $i = 1, \ldots \rho$. Since $\dot{V} \in L_\infty$, it can be concluded from Babalat's lemma [11] $\lim_{t\to\infty} V(t) = 0$, which further implies $\lim_{t\to\infty}\tilde{x}(t) = 0$, and $\lim_{t\to\infty} z_i(t) = 0$. The boundedness of y and $\lim_{t\to\infty} y(t) = 0$ follow the results of z_i with $i = 1$. The boundedness of other state variables can be established using the boundedness of y and the minimum phase assumption in Assumption 3. This concludes the proof of the theorem.

If the system (1) is linear, a simpler control design without invoking backstepping and Assumption 3, can be proposed. For the linear system, the term relating to the output is expressed by $\psi(y) = fy$, with $f \in R^n$. In this case, the following control design is proposed

$$u_l = k_l^T\hat{x} + \hat{w} \quad (73)$$

where k_l is chosen so that $A_l = (A_c + fC + bk_l^T)$ is Hurwitz.

Corollary 5.2 For a system (1) satisfying Assumptions 1, and 2, the control input u shown in (73) stabilizes the system (1) and completely rejects the unknown disturbance if $\phi(y) = fy$.

Remark 4: The results described in Theorems 5.1 and 5.2 are for the general periodic disturbances. For the cases of the square wave and triangular wave disturbances described in Corollaries 4.2 and 4.3, the results shown in Theorems 5.1 and Corollary 5.2 hold without the requirement of Assumption 2, with Assumption 1 automatically satisfied.

VI. An example

Consider a nonlinear system in output feedback form

$$\begin{aligned} \dot{x}_1 &= x_2 - y^3 + (u - w) \\ \dot{x}_2 &= (u - w) \\ y &= x_1 \end{aligned} \tag{74}$$

It is easy to see that the system (74) are in the format of (1) with $\phi(y) = [y^3\ 0]^T$ and $b = [1\ 1]^T$. The system is minimum phase, and therefore Assumption 3 is satisfied. Following the control design introduced in Section 4, the control input is designed as

$$u = \hat{w} - c_1 y - d_1 y - y^3 - \hat{x}_2 \tag{75}$$

The simulation study has been carried out for the estimation and control design shown in this example. In the simulation study, the control design are set as $k_1 = -2$, $k_2 = -1$, $c_1 = d_1 = \lambda_1 = 1$. Three different disturbances used in simulation are sinusoidal disturbance, square wave disturbance and triangular wave disturbance. The settings for all the disturbances are with the period $T = 2$, and the amplitude $a = 1$. For the square eave disturbance, the control input and the system output are shown in Figure 1, in which the output converges to zero with the input to asymptotically cancel the disturbance.

VII. Conclusions

In this paper, we have proposed novel algorithms for asymptotic rejection of general disturbances. The proposed estimation algorithms are able to deal with symmetric smooth disturbances such as sinusoidal disturbances as well as some non-smooth disturbances like square wave and triangular wave disturbances when the periods of the disturbances are known. The success in the disturbance estimation replies on the introduction of the half-period integration, which keeps the desired properties in the wave form for manipulation. A complete set of results relating to the use of half-period integration with delay operators are presented for disturbance estimation for a linear dynamic system with the disturbance as the input, and this result is then applied to estimation of disturbances in the nonlinear systems in output feedback form. With the disturbance estimated, the control design is then proposed for asymptotic rejection of general period disturbances in the systems considered. The proposed estimation algorithms guarantee the convergent estimates in the sense of L_p, and the proposed disturbance rejection algorithms guarantee the asymptotic disturbance rejection with boundedness of all the other variables.

The results presented in this paper extends the class of disturbances which can be asymptotically rejected from sinusoidal disturbances to general periodic disturbances, and the asymptotic rejection does not rely on the conventional internal model principle. The proposed algorithms exploit the wave forms of periodic disturbances, and it can be viewed as an alternative to the methods based on the internal model principle for asymptotic rejection. Future research can be expected in generalizing the results to other nonlinear systems and tackling the disturbances with unknown periods.

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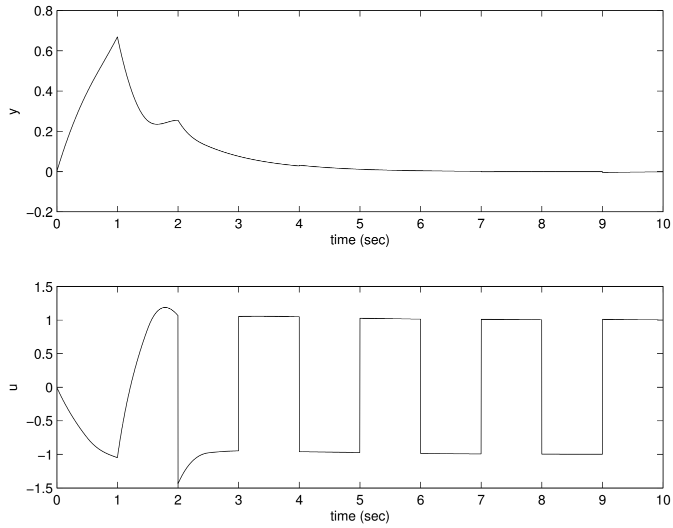

Fig. 1. The system input and output under square wave disturbance