Topology Feedback Quiz, week 5: Hausdorffness and compactness

Open books. 10–15 minutes. Not for credit. To be marked in class.

Convention: $\mathbb{R}$ denotes the **Euclidean (metric) space** unless another topology is given.

Metric topology is Hausdorff.

A subspace of a Hausdorff space is Hausdorff so all subspaces of $\mathbb{R}$ are Hausdorff — we cancel all non-Hausdorff options in the first 5 cases below. ~~non-Hausdorff~~

Question 1 ♣ The closed bounded interval $[10, 20]$ in $\mathbb{R}$ is:

● Hausdorff compact ○ Hausd. non-comp. ○ ~~non-Haus. comp.~~ ○ ~~non-H. non-comp.~~

Explanation:

By the **Heine-Borel Lemma**, $[0, 1]$ is compact.

Every **closed bounded interval** $[a, b]$ with $a < b$ is **homeomorphic to** $[0, 1]$: e.g., the linear function $h\colon [0, 1] \to [a, b]$, $h(x) = a + (b - a)x$, is a homeomorphism.

Compactness is a topological property, so every **closed bounded interval is compact.**

The half-open interval $[0, 1)$ in $\mathbb{R}$ is

○ Hausdorff compact ● Hausd. non-comp. ○ ~~non-Haus. comp.~~ ○ ~~non-H. non-comp.~~

Explanation:

In a Hausdorff space, a compact is closed. $[0, 1)$ is **not closed** in $\mathbb{R}$ so not compact.

The union $[0, 1/3] \cup [2/3, 1]$ is

● Hausdorff compact ○ Hausd. non-comp. ○ ~~non-Haus. comp.~~ ○ ~~non-H. non-comp.~~

Explanation:

A closed subset of a compact is compact. $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ is closed in the compact $[0, 1]$.

The set $\bigcup_{n=1}^{\infty} [\frac{1}{2n+1}, \frac{1}{2n}]$ is

○ Hausdorff compact ● Hausd. non-comp. ○ ~~non-Haus. comp.~~ ○ ~~non-H. non-comp.~~

Explanation:

In a Hausdorff space, a compact is closed. The set $X = \bigcup_{n=1}^{\infty} [\frac{1}{2n+1}, \frac{1}{2n}]$ is **not closed** in $\mathbb{R}$ because $\mathbb{R} \setminus X$ is not open.

Indeed, $0 \in \mathbb{R} \setminus X$, yet no open interval $B_\varepsilon(0) = (-\varepsilon, \varepsilon)$ is contained in $\mathbb{R} \setminus X$: this interval contains $\frac{1}{2n}$ where $\frac{1}{2n} < \varepsilon$, and $\frac{1}{2n} \in X$.

The set $\{0\} \cup \bigcup_{n=1}^{\infty} [\frac{1}{2n+1}, \frac{1}{2n}]$ is

● Hausdorff compact ○ Hausd. non-comp. ○ ~~non-Haus. comp.~~ ○ ~~non-H. non-comp.~~

Explanation:

A closed subset of a compact is compact. The given set X is closed in the compact $[0,1]$. (Note that, unlike in the previous example, $0 \in X$.) Indeed, $\mathbb{R} \setminus X = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} (\frac{1}{2n+2}, \frac{1}{2n+1}) \cup (\frac{1}{2}, +\infty)$ is open as a union of open intervals.

$\mathbb{R}$ with cofinite topology is

○ Hausdorff compact ○ Hausd. non-comp. ● non-Haus. comp. ○ non-H. non-comp.

Explanation:

A **cofinite topology** on $\mathbb{R}$ is **not Hausdorff** because two non-empty open sets U, V cannot be disjoint.

Indeed, $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ must be finite; by the De Morgan laws, $\mathbb{R} \setminus (U \cap V) = \mathbb{R} \setminus U \cup \mathbb{R} \setminus V$.

This set is finite and so cannot be $\mathbb{R}$, meaning that $U \cap V$ cannot be empty. [*Same proof as Example 11.6 in* Sutherland]

We show that the **cofinite topology** is **compact.** Let $\mathcal{C}$ be an open cover of ($\mathbb{R}$, cofinite), and pick a non-empty set $U \in \mathcal{C}$. Since U is open, by definition of cofinite topology

$$U = \mathbb{R} \setminus \{x_1, x_2, \dots, x_n\}$$

for some finite subset $\{x_1, x_2, \dots, x_n\}$ of $\mathbb{R}$. Since $\mathcal{C}$ covers $\mathbb{R}$, in particular $\mathcal{C}$ must cover each of the x_i, so there are sets $V_1, \dots, V_n \in \mathcal{C}$ such that $V_i \ni x_i$ for all i. Then $U \cup V_1 \cup \dots \cup V_n$ contains U and the points $x_1, \dots, x_n$, hence is the whole of $\mathbb{R}$. Thus,

$$U, V_1, \dots, V_n$$

is a finite subcover of $\mathcal{C}$. Compactness is proved. [*Same as Example 13.8 in* Sutherland]

$\mathbb{R}$ with antidiscrete topology is

○ Hausdorff compact ○ Hausd. non-comp. ● non-Haus. comp. ○ non-H. non-comp.

Explanation:

You will see in question 2 below that

a topology is non-Hausdorff ⇒ all weaker topologies are also non-Hausdorff;
a topology is compact ⇒ all weaker topologies are also compact.

Since ($\mathbb{R}$, cofinite) is non-Hausdorff compact and the antidiscrete topology is weaker than cofinite, we conclude that ($\mathbb{R}$, antidiscrete) is also non-Hausdorff compact.

Question 2 ♣ Suppose T_{weak}, T_{strong} are topologies on a set X such that T_{strong} is stronger than T_{weak}. What must be true?

● the function $id_X : (X, T_{strong}) \to (X, T_{weak})$ is continuous

Explanation:

Let U be open in T_{weak}. Note that $id_X^{-1}(U) = U$ and that U is open in T_{strong}: indeed, everything open in T_{weak} is open in T_{strong}. This proves that the preimage of any open set is open, so the function id_X is continuous.

○ the function $id_X : (X, T_{strong}) \to (X, T_{weak})$ is a homeomorphism

Explanation:

No, the inverse function $id_X^{-1} : (X, T_{weak}) \to (X, T_{strong})$ may not be continuous: if T_{strong} is strictly stronger, then there is $V \subset X$ such that V is open in T_{strong} but V (which is its own preimage) is not open in T_{weak}.

● T_{weak} is Hausdorff implies T_{strong} is Hausdorff

Explanation: This was shown in lectures.

● T_{strong} is compact implies T_{weak} is compact

Explanation:

By the first part of the question, (X, T_{weak}) is a continuous image of (X, T_{strong}), and **a continuous image of a compact is compact.** [Sutherland, *Exercise 13.5*]