Topology Feedback Quiz, week 2: bases, continuous functions

Open books. 10–15 minutes. Not for credit. To be marked in class.

We will consider three topological spaces:

- $\mathbb{R}_{\text{antidiscrete}}$, the real line with antidiscrete topology **(RECALL: only $\emptyset$ and $\mathbb{R}$ are open)**
- $\mathbb{R}$, the real line with Euclidean topology **(RECALL: open set = union of open intervals)**
- $\mathbb{R}_{\text{discrete}}$, the real line with discrete topology **(RECALL: <u>all</u> subsets of $\mathbb{R}$ are open)**

RECALL:

— An **open cover** consists of **open** sets whose union is $\mathbb{R}$;

— a **base** is an open cover such that every open set is a union of some sets from the base.

Question 1 ♣ Is the collection

$$\{(a, b) : a, b \in \mathbb{R}\}$$

of all intervals a base, or at least an open cover, for each of the three spaces?

○ base for $\mathbb{R}_{\text{antidiscrete}}$

○ open cover for $\mathbb{R}_{\text{antidiscrete}}$

Explanation: the interval (a, b) is **not open** in $\mathbb{R}_{\text{antidiscrete}}$ so this collection is not an open cover, hence not a base.

● base for $\mathbb{R}$

Explanation: Open intervals are the Euclidean metric balls in $\mathbb{R}$ and form a base of the Euclidean topology by definition.

● open cover for $\mathbb{R}$

Explanation: The open intervals are open in the Euclidean topology and cover all of $\mathbb{R}$.

○ base for $\mathbb{R}_{\text{discrete}}$

Explanation: Take the open set $\{1\}$ in $\mathbb{R}_{\text{discrete}}$. This set is **not** a union of intervals of the form (a, b) so we do not have a base.

● open cover for $\mathbb{R}_{\text{discrete}}$

Explanation: The intervals are open in the discrete topology — because all sets are open — and cover all of $\mathbb{R}$.

Question 2 ♣ Is the collection

$$\{\{p\} : p \in \mathbb{R}\}$$

of all **singletons** a base, or at least an open cover, for each of the three spaces?

○ base for $\mathbb{R}_{\text{antidiscrete}}$

○ open cover for $\mathbb{R}_{\text{antidiscrete}}$

Explanation: A singleton set $\{p\}$ is not open in $\mathbb{R}_{\text{antidiscrete}}$ so this collection is **not** an open cover and not a base.

○ base for $\mathbb{R}$

○ open cover for $\mathbb{R}$

Explanation: A singleton set $\{p\}$ is not open in the Euclidean topology so this collection is **not** an open cover and not a base.

● base for $\mathbb{R}_{\text{discrete}}$

Explanation: An open set is just an arbitrary subset of $\mathbb{R}_{\text{discrete}}$, and surely all sets can be written as unions of singletons, so singletons form a base of the discrete topology.

● open cover for $\mathbb{R}_{\text{discrete}}$

Explanation: Singletons are open in $\mathbb{R}_{\text{discrete}}$ and cover the whole of $\mathbb{R}_{\text{discrete}}$.

Question 3 ♣ Which functions on $\mathbb{R}^2$ are continuous? Here $\mathbb{R}^2$ has Euclidean topology, and (x, y) denotes a point in $\mathbb{R}^2$.

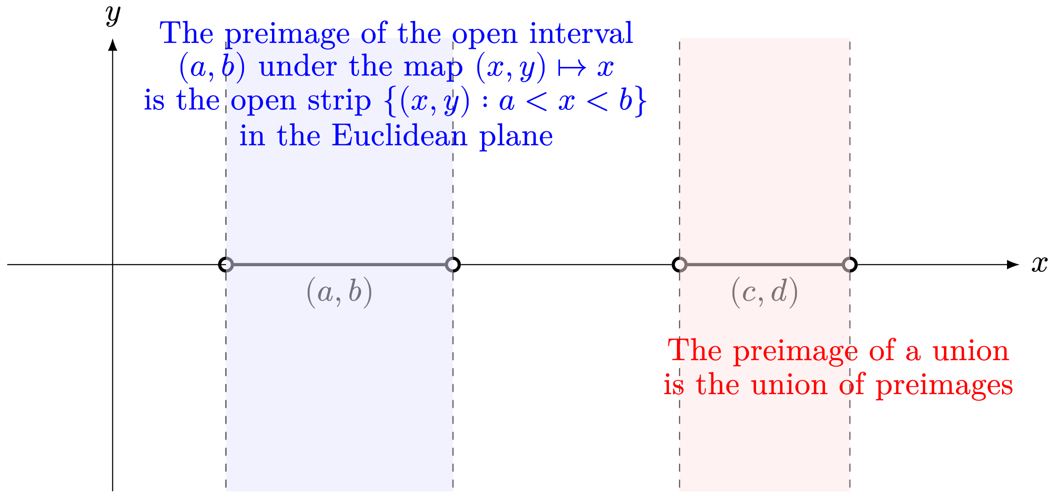
● $\mathbb{R}^2 \to \mathbb{R}_{\text{antidiscrete}}$, $(x, y) \mapsto x$

Explanation: We prove that the preimage of every open set is open. There are two open sets in $\mathbb{R}_{\text{antidiscrete}}$: $\emptyset$ and $\mathbb{R}$. We have $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\mathbb{R}) = \mathbb{R}^2$. In both cases the preimage is open in $\mathbb{R}^2$, so f is continuous.

● $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$

Explanation: Since we are dealing here with a function between two metric topologies, we can use methods from Metric Spaces — for example the ε - δ definition — to show that the function is continuous.

Yet here we can also directly verify the topological definition of continuity. Indeed, the preimage of an open interval $(a, b) \subseteq \mathbb{R}$ is the "open strip" $\{(x, y) : a < x < b\}$ in $\mathbb{R}^2$, and so the preimage of a union of open intervals is a union of open strips, which is open in $\mathbb{R}^2$.



○ $\mathbb{R}^2 \to \mathbb{R}_{\text{discrete}}$, $(x, y) \mapsto x$

Explanation: The singleton set $\{2\}$ is open in $\mathbb{R}_{\text{discrete}}$ — because all sets are open in this topology. Its preimage in $\mathbb{R}^2$ is the vertical line $y = 2$. This line is not an open set in the Euclidean plane. Therefore, the function is not continuous.

Question 4 Write down an example of a **continuous** function $f\colon \mathbb{R}^2 \to \mathbb{R}$ (Euclidean topology on both) and sets $A, B \subset \mathbb{R}^2$ such that:

- A is open but $f(A)$ is not open;
- B is closed but $f(B)$ is not closed.

The function $(x, y) \mapsto x$ will not work here, because — as one can prove — it sends open sets to open sets. We will modify this function. Let

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \qquad f((x,y)) = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

The function f is continuous: this is clear from the formula $f((x,y)) = \dfrac{x + |x|}{2}$.
Put $A = \mathbb{R}^2$, which is certainly open in $\mathbb{R}^2$. Then $f(A) = [0, +\infty)$ is a closed half-line in $\mathbb{R}$. The set $[0, +\infty)$ is **not open** in $\mathbb{R}$.
Let $B = \{(x, \frac{1}{x}) : x > 0\}$ be the positive half of the graph of the function $\frac{1}{x}$. Then B is a closed subset of the plane, yet $f(B) = (0, +\infty)$ is a subset of $\mathbb{R}$ which is **not** closed.