Week 1: model solutions to the extra exercises

These exercises were given in Week 1 but not revisited since.

Exercise 1 below is a nice revision exercise.

Exercise 2 below is beyond the scope of the course and has little to do with topology: a non-trivial exercise in real analysis and number theory, it can be safely ignored when revising MATH31010.

Exercise 1. Call a subset A of $\mathbb{R}$ "cocountable" if $A = \emptyset$ or $\mathbb{R} \setminus A$ is finite or countably infinite.

(a) Show that the collection of all cocountable subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$.

(b) Is this topology the same as discrete topology? Antidiscrete topology? Cofinite topology?

Solution. (a) The proof that the collection of cocountable subsets of $\mathbb{R}$ is a topology is the same as the proof for cofinite topology, but the word "finite" must be replaced by the word **countable** (meaning: finite or countably infinite) throughout. Let us do this exercise.

Proposition (the cocountable topology is a topology). Let X be a set. The collection $\mathscr{C}$ which consists of the empty set and all subsets of X with countable complement is a topology on the set X.

Proof. We show that $\mathscr{C}$ satisfies axioms (i)–(iii) from the definition of topology.

(i) X has complement $\emptyset$, and $\emptyset$ is **countable**, so $X \in \mathscr{C}$.

(ii) Let $\mathscr{F}$ be some collection of sets from $\mathscr{C}$. If all sets in $\mathscr{F}$ are empty, then $\bigcup \mathscr{F} = \emptyset \in \mathscr{C}$.

Otherwise, take a non-empty set $U \in \mathscr{F}$. Then U must have **countable** complement, and $U \subseteq \bigcup \mathscr{F}$. Since taking comlement reverses inclusion, $X \setminus \bigcup \mathscr{F} \subseteq X \setminus U$. Yet $X \setminus U$ is a **countable** set, and all subsets of a **countable** set are **countable**. Hence the complement of $\bigcup \mathscr{F}$ is **countable**, proving that $\bigcup \mathscr{F}$ is in $\mathscr{C}$.

(iii) Suppose $U, V \in \mathscr{C}$. If one of U, V is an empty set, then $U \cap V = \emptyset \in \mathscr{C}$.

Otherwise, U and V are non-empty, and since they are in $\mathscr{C}$, U and V must have **countable** complements. Then by the De Morgan laws $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$. Thus, $U \cap V$ has a **countable** complement (a union of two **countable** sets) and so $U \cap V \in \mathscr{C}$. □

(b) The "cocountable" topology on $\mathbb{R}$

- is not discrete: the set $\{1\}$ is open in discrete but not open in cocountable;
- is not antidiscrete: the set $\mathbb{R} \setminus \{1\}$ is open in cocountable but is not open in antidiscrete;
- is not cofinite: the set $\mathbb{R} \setminus \mathbb{Q}$ is open in cocountable but not open in cofinite.

Exercise 2 (harder). Let

$$A = \bigcap_{n \in \mathbb{N}} \bigcup_{p \in \mathbb{Z},\, q \in \mathbb{N}} \left(\frac{p}{q} - \frac{1}{nq}, \frac{p}{q} + \frac{1}{nq} \right).$$

Denote by $\mathbb{Q}$ the set of all rationals. Is $\mathbb{Q} \subseteq A$? Is $\mathbb{Q} = A$? Is $A = \mathbb{R}$?

Solution. In fact, $A = \mathbb{R}$ which takes just a couple of lines to prove (see Stage 3 below) but requires knowledge beyond basic topology and real analysis. We will approach the question in stages.

Denote by A_n the set $\bigcup_{p\in\mathbb{Z},\, q\in\mathbb{N}} \left(\frac{p}{q} - \frac{1}{nq}, \frac{p}{q} + \frac{1}{nq}\right)$ so that $A = \bigcap_{n\in\mathbb{N}} A_n$.

Stage 1. The easiest observation to make is that $\mathbb{Q} \subseteq A$. Indeed, let $x \in \mathbb{Q}$. Then x can be written as $x = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ one has $x = \frac{p}{q} \in \left(\frac{p}{q} - \frac{1}{nq}, \frac{p}{q} + \frac{1}{nq}\right)$, and so $x \in A_n$. Therefore x lies in the intersection of all the sets A_n, which is A.

Stage 2. We now claim that $\mathbb{Q} \neq A$. We justify this by exhibiting an element of A which is not a rational number. This is a pleasant exercise in undergraduate real analysis.

Consider the irrational number e defined as the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}$. We claim that $e \in A$. Indeed, let s_n be the partial sum $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$. Then s_n is a rational number, as it is a sum of finitely many rationals. Moreover, s_n can be written in the form $\frac{p_n}{q_n}$ where p_n is an integer and $q_n = n!$. We note that

$$\frac{p_n}{q_n} < e < \frac{p_n}{q_n} + \frac{1}{n(n!)}$$

because $e - \frac{p_n}{q_n} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots$ can be bounded from above by $\frac{1}{n(n!)}$ by comparison with the geometric series $\frac{1}{(n+1)!} + \frac{1}{(n+1)!(n+1)} + \frac{1}{(n+1)!(n+1)^2} + \cdots = \frac{1}{n(n!)}$.
This means that $e \in A_n$. Since the argument is valid for all $n \geq 1$, we conclude that $e \in A$.

Stage 3. Finally, we claim (and this is the claim which requires knowledge beyond MATH31010) that $A = \mathbb{R}$. We use Dirichlet's Approximation Theorem (a result in Number Theory):

For any $x \in \mathbb{R}$ and any positive integer n there exist integers p, q with $0 < q \leq n$ such that

$$|qx - p| < \frac{1}{n}.$$

The inequality in Dirichlet's Theorem rewrites as $|x - \frac{p}{q}| < \frac{1}{nq}$, immediately showing that $x \in A_n$. Since the Theorem says "for any positive integer n", we have $x \in A_n$ for all n, hence $x \in A$. Since the Theorem says "for any $x \in \mathbb{R}$", we have proved that $\mathbb{R} \subseteq A$ and so $A = \mathbb{R}$.