Topology Feedback Quiz, week 1: sets and functions

Open books. 10–15 minutes. Not for credit. To be marked in class.

We work with functions $f\colon \mathbb{R} \to \mathbb{R}$.

Recall that the image of a set $A \subseteq R$ under f is $f(A) = \{f(a) : a \in A\}$, and the preimage of a set $B \subseteq \mathbb{R}$ is $f^{-1}(B) = \{x \in \mathbb{R} : f(x) \in B\}$.

"Is it always true that..." below asks whether the statement is true for **all** functions $f\colon \mathbb{R} \to \mathbb{R}$ and **all** subsets $A, B \subseteq \mathbb{R}$.

Question 1 Is it always true that $f(A \cup B) = f(A) \cup f(B)$?

● Yes ○ No, give a counterexample ______________________________

Explanation: **It is important to remember, or be able to work out informally, how the images and preimages behave with respect to set operations.**

It is less important to be able to give a formal proof of each statement about this behavior, but for completeness we do give such (rather long) proofs below.

The first implication that we will prove is $y \in f(A \cup B) \implies y \in f(A) \cup f(B)$.

Suppose that $y \in f(A \cup B)$. By definition of the image of a set, there exists $x \in A \cup B$ such that $f(x) = y$. Take one such x. Since $x \in A \cup B$, by definition of union we have $x \in A$ or $x \in B$. Consider two cases:

- case 1, $x \in A$. Since $y = f(x)$, we have $y \in f(A)$.
- case 2, $x \in B$. Since $y = f(x)$, we have $y \in f(B)$.

In either case, the statement "$y \in f(A)$ OR $y \in f(B)$" is true. By definition of union of sets, this statement is equivalent to $y \in f(A) \cup f(B)$.

The second implication that we will prove is $y \in f(A) \cup f(B) \implies y \in f(A \cup B)$.

Suppose that $y \in f(A) \cup f(B)$. By definition of union, $y \in f(A)$ OR $y \in f(B)$. Consider two cases:

- case 1, $y \in f(A)$. Then $y = f(x)$ for some $x \in A$. Note that $x \in A$ implies $x \in A \cup B$, hence we have shown that $y = f(x) \in f(A \cup B)$.
- case 2, $y \in f(B)$. Then $y = f(x)$ for some $x \in B$. Note that $x \in B$ implies $x \in A \cup B$, hence we have shown that $y = f(x) \in f(A \cup B)$.

In either case, the statement $y \in f(A \cup B)$ is true.

We have proved two implications, which together mean that $y \in f(A \cup B) \iff y \in f(A) \cup f(B)$. This means equality of the two sets: $f(A \cup B) = f(A) \cup f(B)$, q.e.d.

Question 2 Is it always true that $f(A \cap B) = f(A) \cap f(B)$?

○ Yes ● No, give a counterexample $f(x) = 1$ for all x, $A = \mathbb{R}_{<0}$, $B = \mathbb{R}_{>0}$

Explanation: In the counterexample, $A \cap B = \emptyset$, $f(A \cap B) = \emptyset$, yet $f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\}$.

Remark: although $f(A \cap B)$ may fail to be equal to $f(A) \cap f(B)$, we always have $f(A \cap B) \subseteq f(A) \cap f(B)$. To prove this inclusion, let $y \in f(A \cap B)$, and take $x \in A \cap B$ such that $y = f(x)$. Then $x \in A$, so $y \in f(A)$; AND $x \in B$, so $y \in f(B)$. We conclude that $y \in f(A) \cap f(B)$.

Question 3 Is it always true that $f^{-1}(A \cup B) = f(A) \cup f(B)$? corrected: $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

● Yes ○ No, give a counterexample ______

Explanation: $x \in f^{-1}(A \cup B) \iff f(x) \in A \cup B \iff \big(f(x) \in A \text{ OR } f(x) \in B\big) \iff \big(x \in f^{-1}(A)$ OR $x \in f^{-1}(B)\big) \iff x \in f^{-1}(A) \cup f^{-1}(B)$.

Question 4 Is it always true that $f^{-1}(A \cap B) = f(A) \cap f(B)$? corrected: $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$?

● Yes ○ No, give a counterexample ______

Explanation: $x \in f^{-1}(A \cap B) \iff f(x) \in A \cap B \iff \big(f(x) \in A \text{ AND } f(x) \in B\big) \iff \big(x \in f^{-1}(A)$ AND $x \in f^{-1}(B)\big) \iff x \in f^{-1}(A) \cap f^{-1}(B)$.

Question 5 Is it always true that $f(\mathbb{R} \setminus A) = \mathbb{R} \setminus f(A)$?

○ Yes ● No, give a counterexample $f(x) = 1$ for all x, $A = \mathbb{R}$

Explanation: In the counterexample, $f(\mathbb{R} \setminus A) = f(\emptyset) = \emptyset$, yet $\mathbb{R} \setminus f(A) = \mathbb{R} \setminus \{1\} \neq \emptyset$.

Question 6 Is it always true that $f^{-1}(\mathbb{R} \setminus A) = \mathbb{R} \setminus f^{-1}(A)$?

● Yes ○ No, give a counterexample ______

Explanation: $x \in f^{-1}(\mathbb{R} \setminus A) \Leftrightarrow f(x) \in \mathbb{R} \setminus A \Leftrightarrow f(x) \notin A \Leftrightarrow x \notin f^{-1}(A) \Leftrightarrow x \in \mathbb{R} \setminus f^{-1}(A)$.

Question 7 Is it always true that if A is finite, then $f(A)$ is finite?

● Yes ○ No, give a counterexample ______

Explanation: Assume that A is finite and has cardinality n. Then $A = \{x_1, x_2, \dots, x_n\}$. Then $f(A) = \{f(x_1), f(x_2), \dots, f(x_n)\}$. There may be repetitions in the list $f(x_1), f(x_2), \dots, f(x_n)$, but in any case the cardinality of $f(A)$ is at most n and so is finite.

Question 8 Is it always true that if A is finite, then $f^{-1}(A)$ is finite?

○ Yes ● No, give a counterexample $f(x) = 1$ for all x, $A = \{1\}$ finite, $f^{-1}(A) = \mathbb{R}$ infinite

Question 9 Is it always true that if A is infinite, then $f(A)$ is infinite?

○ Yes ● No, give a counterexample $f(x) = 1$ for all x, $A = \mathbb{R}$ infinite, $f(A) = \{1\}$ finite

Question 10 Is it always true that if A is infinite, then $f^{-1}(A)$ is infinite?

○ Yes ● No, give a counterexample $f(x) = 1$ for all x, $A = \mathbb{R}_{<0}$ infinite, $f^{-1}(A) = \emptyset$ finite

Exercise. Call a subset A of $\mathbb{R}$ "cocountable" if $A = \emptyset$ or $\mathbb{R} \setminus A$ is finite or countably infinite.
(a) Show that the collection of all cocountable subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$.
(b) Is this topology the same as discrete topology? Antidiscrete topology? Cofinite topology?
This exercise will be discussed in the week 2 tutorial.

Exercise (harder). Let
$$A = \bigcap_{n\in\mathbb{N}} \bigcup_{p\in\mathbb{Z},\ q\in\mathbb{N}} \left(\frac{p}{q} - \frac{1}{nq}, \frac{p}{q} + \frac{1}{nq}\right).$$
Denote by $\mathbb{Q}$ the set of all rationals.
Is $\mathbb{Q} \subseteq A$? Is $\mathbb{Q} = A$? Is $A = \mathbb{R}$?
This exercise will not be discussed in class, but a solution will be made available later.