Week 9 tutorial

Exercise 9.0. An unseen exercise on gluing. Definition: let X be a top. space, Y a set (initially without a topology), $q\colon X \to Y$ a **surjective** map. The topology on Y given by

$$V \subseteq Y \text{ is open in } Y \overset{\text{def}}{\iff} q^{-1}(V) \text{ is open in } X$$

is **the quotient topology** on Y induced by q. In this situation, q is **the quotient map.**

In particular, if $\sim$ is an equivalence relation on X, take $Y := X/\sim$ to be the set of $\sim$-equivalence classes, and define $q\colon X \to Y$ by $q(x) = [x]$. The quotient topology on $X/\sim$ is called **the identification topology** with respect to $\sim$. (Idea: whenever $x' \sim x''$, we identify points x' and x'' and treat them as one point.)

Gluing topology is identification topology where $\sim$ is such that

- some equivalence classes consist of 2 points (two points **glued together**);
- the rest of equivalence classes are singletons.

Theorem (universal mapping property for a quotient space). Let Y be a quotient space via the quotient map $X \xrightarrow{q} Y$. Given any topological space Z, there is a 1-to-1 correspondence between

- continuous maps $f\colon Y \to Z$;
- continuous $F\colon X \to Z$ such that $F(x') = F(x'')$ in Z whenever $q(x') = q(x'')$ in Y.

The correspondence is such that $q \circ F = f$.

CHALLENGE: construct embeddings, or at least "immersions", in $\mathbb{R}^3$, of the gluing spaces given by schematic diagrams presented in class.

$0 \sim 1 : [0] = [1] = \{0,1\}$

$\sim : \forall x \neq 0,1,\ [x] = \{x\}$

$X =$ 

$X/\sim$: 

Meaning of the UMP : continuous functions from $X/\sim$ to some space Z $\overset{1:1}{\longleftrightarrow}$ functions $F: X \to Z$ which respect the gluing $\sim$, i.e. whenever $x' \sim x''$, we have $F(x') = F(x'')$.

So To show $[0,1]/\sim \;\cong S^1$, need a cont. function $F: [0,1] \to S^1$, $F(0) = F(1)$ and is a bijection $[0,1]/\sim \;\longleftrightarrow S^1$.

<u>Note</u> $[0,1]$ is compact (HB lemma)

$\Rightarrow [0,1]/\sim$ is also compact because the quotient map $[0,1] \longrightarrow [0,1]/\sim$ is continuous (and a cont. image of compact is compact)

$$\underset{\text{compact}}{[0,1]/\sim} \xrightarrow[\text{bijection}]{f \text{ continuous}} \underset{\text{metric} \Rightarrow \text{Hausdorff}}{S^1}$$

by the JIFT, f is a homeomorphism.

$F : [0,1] \longrightarrow S^1$ $\quad F(0) = F(1)$

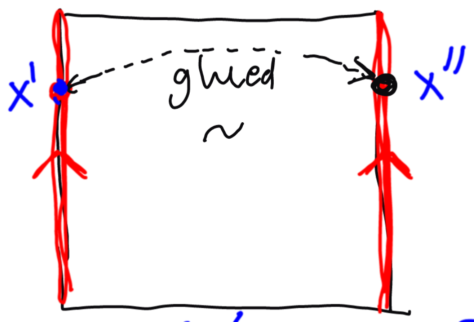
$$F(t) = (\cos(2\pi t), \sin(2\pi t))$$

Call $[0,1]/\sim$: $0 \sim 1$ nothing else is glued — the "abstract circle"

We have embedded the abstract circle in $\mathbb{R}^2$.

<u>Next example</u>

$X = [0,1] \times [0,1]$



"abstract cylinder"

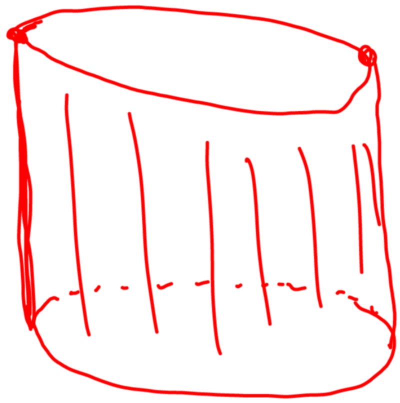
Embed the space $X/\sim$ in $\mathbb{R}^3$:

$$F : [0,1] \times [0,1] \xrightarrow{\text{cont.}} \mathbb{R}^3$$

$$F((0,y)) = F((1,y)) \ \forall y \in [0,1]$$

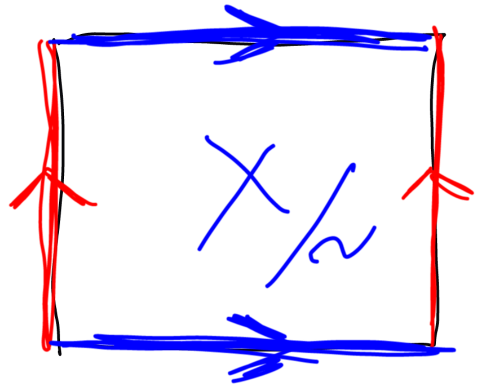
F is bijective on $X/\sim$

E.g. put $F((x,y)) = (\cos(2\pi x), \sin(2\pi x), y) \in \mathbb{R}^3$



cylinder $S^1 \times [0,1]$ in $\mathbb{R}^3$

Next:



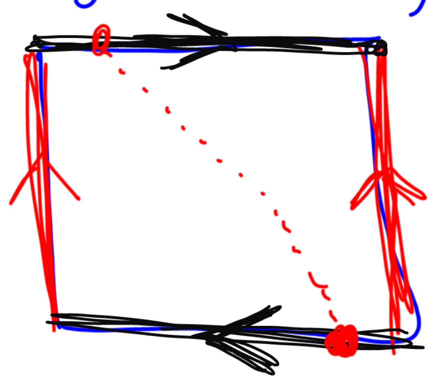
$F : X \longrightarrow \mathbb{R}^3$ such that

$F((0,y)) = F((1,y))$, $F((x,0)) = F((x,1))$

$\left(a + \cos\frac{x}{2}\right)\sin y -$

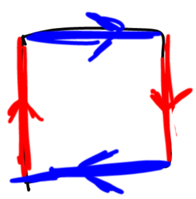
$-\sin\frac{y}{2}\sin 2\ldots$

(see software-generated version)



The abstract Klein bottle

What is



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