Check In

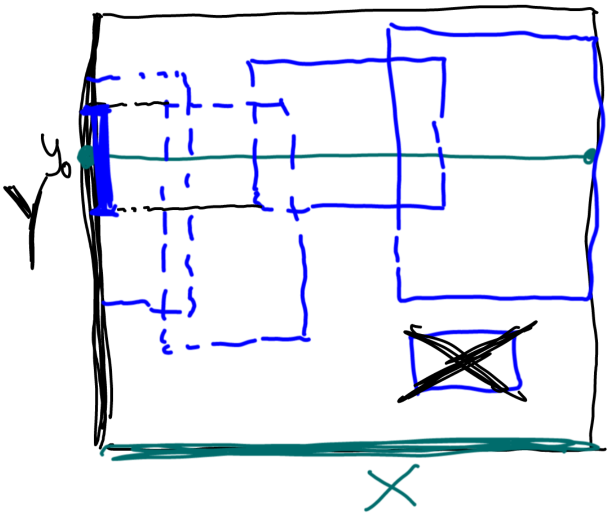
870678

Reminder

- we proved: Lemma: if $\mathcal{B}$ is a base of topology on X and every $\mathcal{B}$-basic cover of X has a finite subcover, then X is compact.
- We will use this to prove

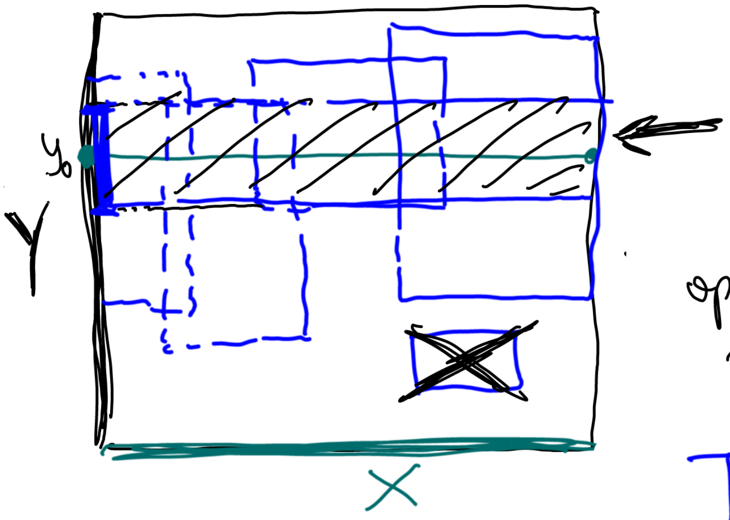
baby Tychonoff: X, Y compact $\Rightarrow$ $X \times Y$ is compact.

(Proof - continued) Assume X, Y are compact. Assume that $\mathcal{C}$ is a cover of $X \times Y$ by open rectangles. We will show: $\mathcal{C}$ has a finite subcover (and so, by the Lemma, $X \times Y$ is compact).



$X \times \{y_0\}$ is homeomorphic to X hence $X \times \{y_0\}$ is a compact subset of $X \times Y$. Therefore, can choose from $\mathcal{C}$ a finite list of open rectangles which covers $X \times \{y_0\}$. [Delete from this finite list any rectangle which does not intersect $X \times \{y_0\}$]

If $U_1 \times V_1, \ldots, U_n \times V_n$ is our list of rectangles, and $V_i \ni y_0$ for all $i = 1, \ldots, n$, put $V(y_0) =$
$= V_1 \cap V_2 \cap \cdots \cap V_n \longleftarrow$ an open nbhd of y_0

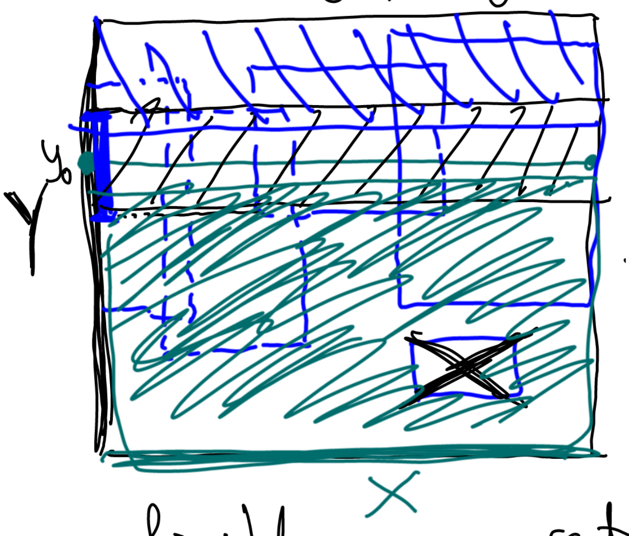


The cylinder set $X \times V(y_0)$ is covered by the open rectangles $U_1 \times V_1, \dots, U_n \times V_n$

Thus, each point $y_0 \in Y$ is covered by an open set $V(y_0)$ constructed above.

$\left.\begin{array}{l} \{V(y_0)\}_{y_0 \in Y} \text{ is an open cover of } Y \\ Y \text{ is a compact space} \end{array}\right\} \Rightarrow$

Y is covered by finitely many nbhds $V(y_1), V(y_2), \dots, V(y_m)$



Then $X \times Y$ is covered by finitely many open cylinders $X \times V(y_1), \dots, X \times V(y_m)$.

By construction, each $X \times V(y_i)$ is covered by finitely many rectangles from $\mathcal{C}$. All these (finitely many) rectangles form a subcover chosen from $\mathcal{C}$ for $X \times Y$. ∎

<u>Corollary</u> $X_1, \dots, X_n$ compact $\Rightarrow$ $X_1 \times X_2 \times \dots \times X_n$ is compact

(understood as $(((X_1 \times X_2) \times X_3) \times X_4) \cdots$)

THM (the Heine-Borel THM) In the Euclidean space $\mathbb{R}^n$, a set K is compact $\iff$ K is closed and bounded.

Pf ($\Rightarrow$) True in ALL metric spaces (proved earlier)

($\Leftarrow$) K is closed and bounded in $\mathbb{R}^n$

$\Rightarrow$ $K \subseteq [-M, M]^n$ for some $M > 0$

$= \{ (x_1, \dots, x_n) : |x_i| \leq M, \ \forall i \}$

and K is closed in $[-M, M]^n$.

$[-M, M]$ is homeomorphic to $[0,1]$ hence compact.

By the baby Tychonoff (and corollary), $[-M, M]^n$ is compact

So K is a closed subset of a compact

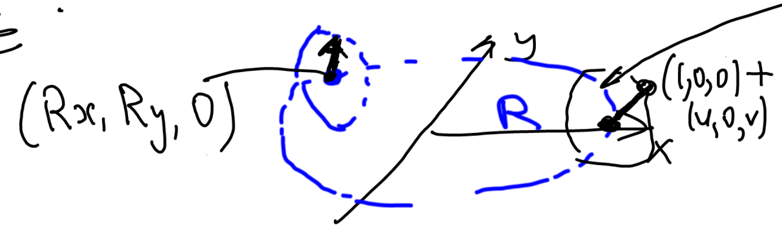
$\Rightarrow$ by an earlier result, K is compact. ∎

Ex $\mathbb{T}^2 = S^1 \times S^1$, S^1 = unit circle in $\mathbb{R}^2$

Since S^1 is closed & bdd in $\mathbb{R}^2$, by HB S^1 is compact $\Rightarrow$ by baby T., $\mathbb{T}^2$ is compact.

Naturally, $\mathbb{T}^2 \subseteq \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4$ $S^1 \times S^1$

Prove :



$f: S^1 \times S^1 \to \mathbb{R}^3$

Want f to be

an <u>embedding</u> (i.e. $f: S^1 \times S^1 \longrightarrow f(S^1 \times S^1)$ is a homeomorphism)

Define $f\left(\left((x,y),(u,v)\right)\right) = (Rx, Ry, 0) +$ Rotation around z axis of $(u, 0, v)$ through the angle θ: $\cos\theta = x$, $\sin\theta = y$

$x^2+y^2 = u^2+v^2 = 1$

To prove that $f: \mathbb{T}^2 \longrightarrow f(\mathbb{T}^2)$ is a homeomorphism, we show:

1. f is continuous: f_x, f_y, f_z are continuous as combinations of the two projections of $\mathbb{T}^2$ on S^1, S^1
2. f is injective (set $R > 1$) and so $f: \mathbb{T}^2 \longrightarrow f(\mathbb{T}^2)$ bijective
3. As f: compact $\longrightarrow$ Hausdorff, f is homeo' by TIFT.