MATH31010 TOPOLOGY AND ANALYSIS **Week** 09 **lecture** A

Check In

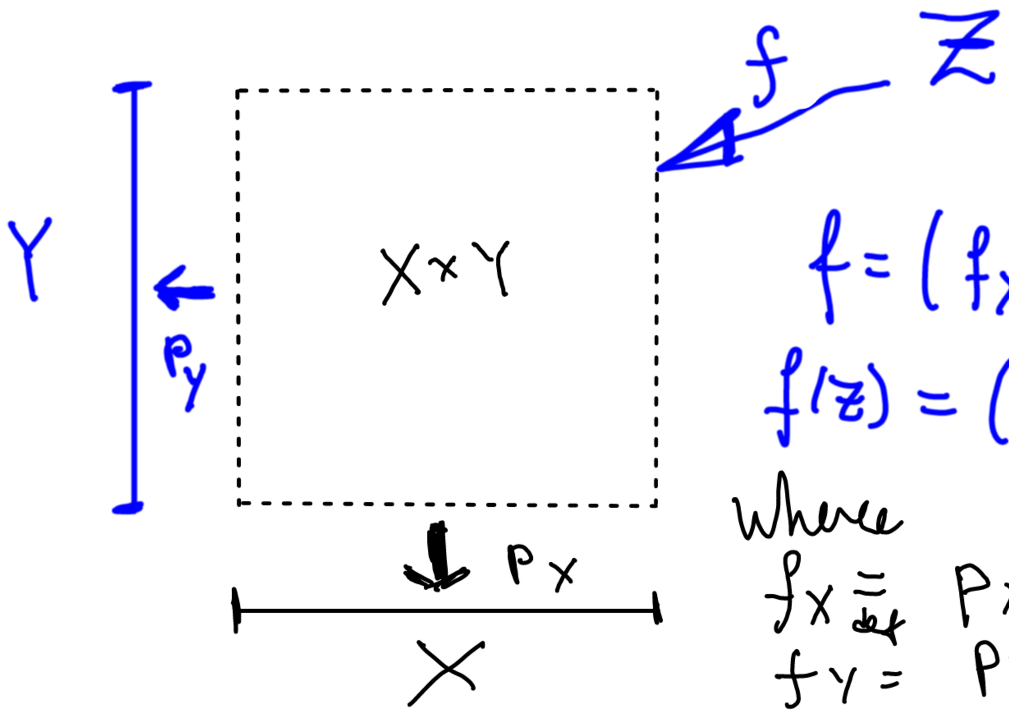
431043

$p_X, p_Y$ are continuous

# Reminder

$X, Y$ top. spaces $\rightsquigarrow$ $X \times Y$ product space

base $\mathcal{B} = \{$open rectangles in $X \times Y\}$




$f = (f_X, f_Y)$

$f(z) = (f_X(z), f_Y(z))$

where

$f_X \underset{\text{def}}{=} p_X \circ f$

$f_Y = p_Y \circ f$

# THM (the Universal Mapping Property of $X \times Y$)

$$\begin{tikzcd} Y & X \times Y \arrow[l, "p_Y"'] \arrow[d, "p_X"] & Z \arrow[l, "f"'] \\ & X & \end{tikzcd}$$

$f(z) = (f_X(z), f_Y(z))$

Given a topological space $Z$,

$f \in$ {continuous maps $f : Z \to X \times Y$}

$\updownarrow$ 1-to-1

$f_X = p_X \circ f$, $f_Y = p_Y \circ f$

{pairs $(f_X, f_Y)$ : $f_X : Z \to X$, $f_Y : Z \to Y$, both continuous}

$(f_X, f_Y)$

**Pf** ① If $f: Z \to X \times Y$ is continuous, then $f_X = p_X \circ f$ is continuous (composition of continuous maps) and $f_Y = p_Y \circ f$ is continuous.

② If I have continuous maps $f_X : Z \to X$, $f_Y : Z \to Y$

and I define $f: Z \to X \times Y$ by $f(z) = (f_X(z), f_Y(z))$, I need to show that $f$ is continuous:

Take an open set in $X \times Y$ : $\bigcup_{\alpha \in I} U_\alpha \times V_\alpha$

where $U_\alpha \underset{\text{open}}{\subseteq} X$, $V_\alpha \underset{\text{open}}{\subseteq} Y$.

$$f^{-1}\Big( \bigcup_\alpha U_\alpha \times V_\alpha \Big) = \bigcup_\alpha f^{-1}(U_\alpha \times V_\alpha)$$

$$f^{-1}(U_\alpha \times V_\alpha) = \{ z \in Z : f_X(z) \in U_\alpha \text{ and } f_Y(z) \in V_\alpha \}$$
$$= \underbrace{f_X^{-1}(U_\alpha)}_{\text{open}} \cap \underbrace{f_Y^{-1}(V_\alpha)}_{\text{open}} = \text{open in } Z.$$

as $f_X$, $f_Y$ are continuous.

We have verified the def$^n$ of "continuous" for $f$.

(3) The correspondences

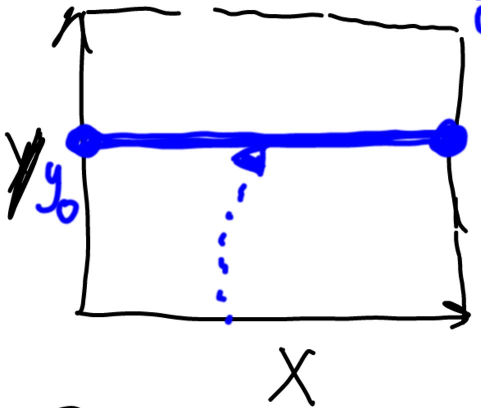
$$f \longmapsto (f_X = p_X \circ f,\ f_Y = p_Y \circ f)$$

and

$$f \longleftarrow\!\!\!\shortmid (f_X, f_Y)$$

are mutually inverse, so we have a 1-to-1 correspondence, as claimed. $\square$

**Prop** Fix $y_0 \in Y$. The map



$$i_{y_0} : X \longrightarrow X \times Y \quad \text{is continuous;}$$
$$x \longmapsto (x, y_0)$$

and is a homeomorphism between $X$ and the set $X \times \{y_0\}$.

**Pf** (1) $i_{y_0}$ is continuous: use UMP.

$$\mathrm{id}_X = p_X \circ i_{y_0} : x \mapsto (x, y_0) \mapsto x$$
continuous (identity map)

$$\mathrm{Const}_{y_0} = p_Y \circ i_{y_0} : x \mapsto (x, y_0) \mapsto y_0$$
continuous (constant map)

so by UMP, $i_0$ is continuous.

② $p_X\big|_{X\times\{y_0\}} : (x, y_0) \longmapsto x$

$X\times\{y_0\} \longrightarrow X$

$$p_X\big|_{X\times\{y_0\}} \circ i_{y_0} = id_X$$

$$\underbrace{(x,y_0)}_{} \overset{p_X}{\longmapsto} x \overset{i_{y_0}}{\longmapsto} \underbrace{(x,y_0)}_{}$$

so $i_{y_0} \circ p_X\big|_{X\times\{y_0\}} = id_{X\times\{y_0\}}$

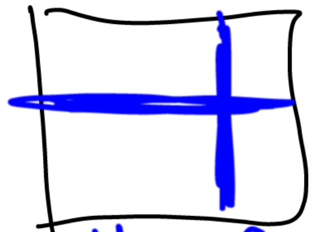
So $p_X\big|_{X\times\{y_0\}}$ is the inverse of $i_{y_0}$. $p_X$ is cont.

$\Rightarrow p_X\big|_{X\times\{y_0\}}$ is continuous (restriction) so homeo'm proved. $\blacksquare$

Skip: * $X, Y$ are both Hausdorff $\iff$ $X \times Y$ is Hausdorff
$X, Y \neq \emptyset$

* $X, Y \neq \emptyset$; $X, Y$ connected $\iff$ $X \times Y$ is connected

"TIKHONOV"



THM (the "baby Tychonoff theorem")

$X, Y \neq \emptyset$: $X, Y$ are compact $\iff$ $X \times Y$ is compact.

Pf $\Leftarrow$: $X = p_X(X \times Y)$
$Y = p_Y(X \times Y)$

$p_X, p_Y$ are continuous, and a continuous image of a compact is a compact.

$\Rightarrow$ : we need:

<u>LEMMA</u> Suppose $\mathcal{B}$ is a base of topology on $X$. Suppose every cover of $X$ by sets from $\mathcal{B}$ has a finite subcover. Then $X$ is compact.

<u>Pf</u> Let $\mathcal{C}$ be a cover of $X$ by arbitrary open sets (not necessarily basic). We need to show that $\mathcal{C}$ has a finite subcover (and then we are done). Each $\mathcal{U} \in \mathcal{C}$ is a union of sets from $\mathcal{B}$: call them "children" of $\mathcal{U}$, so that $\mathcal{U} = \bigcup \{\text{children of } \mathcal{U}\}$

(all children are members of $\mathcal{B}$)

Consider a new open cover:

$$\mathcal{C}_1 = \{\text{all children of all sets from } \mathcal{C}\}$$

Clearly, $\bigcup \mathcal{C}_1 = \bigcup \mathcal{C} = X$ so $\mathcal{C}_1$ is an open cover of $X$ by sets from $\mathcal{B}$.

$\Rightarrow$ by assumption, $\exists$ finite subcover

$$V_1, \ldots, V_n : \quad V_i \in \mathcal{C}_1, \; X = V_1 \cup \ldots \cup V_n$$

Then a parent$(V_1)$, a parent$(V_2)$, ..., a parent$(V_n)$ is a finite subcollection of $\mathcal{C}$;

$$\text{a parent of } V_i \supseteq V_i;$$

$\Rightarrow$ these parents cover the whole $X$, as required. $\blacksquare$

(Proof of Baby Tychonoff: next lecture)

Preview of an example:

$$S^1 = \{(x,y) \in \mathbb{R}^2_{\text{Euclidean}} : x^2+y^2=1\}$$

unit circle.

A compact space.

$$\mathbb{T}^2 \overset{\text{def}}{=} S^1 \times S^1$$ the torus (abstract)

naturally sits in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$

Construct an embedding (i.e. homeo'm to the image)

$$\mathbb{T}^2 \hookrightarrow \mathbb{R}^3$$