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Week 8 tutorial

Exercise 8.0. This is an unseen exercise on closure, boundary and dense sets. Consider the sets $A = \{0, 1\} \subset \mathbb{R}$ and $B = \mathbb{R} \setminus A = (-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ as a subsets of four different topological spaces, given in the table below. Complete the table.

$\{0\}$ is closed in Euclidean topology? Yes: in Hausd., a point is closed. $A = \{0\} \cup \{1\}$ closed in Euclidean

	The space X			
	($\mathbb{R}$, antidiscrete) closed sets: $\{\emptyset, \mathbb{R}\}$	($\mathbb{R}$, cofinite) Closed sets $\mathbb{R}$ and finite sets	($\mathbb{R}$, Euclidean)	($\mathbb{R}$, discrete) all sets are open and closed
$\overline{A}$ (closure in X)	$\mathbb{R}$	$\overline{A} = A$	$\overline{A} = A$	A, because A is closed
Is A dense in X? (yes/no)	Yes	No	No, as $A \neq \mathbb{R}$	(NO)
$\overline{B}$	$\mathbb{R}$	$\mathbb{R}$	$0 \in \overline{B}$, $1 \in \overline{B}$ So $\overline{B} = \mathbb{R}$	B
Is B dense in X? (yes/no)	yes	yes	yes	(NO)
∂A	$\mathbb{R} \cap \mathbb{R} = \mathbb{R}$	A	$\overline{A} \cap \overline{B} = A$	$A \cap B = \emptyset$

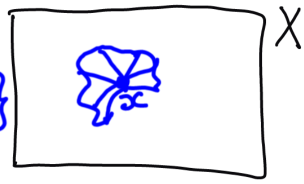
Hint. You may wish to recall that

$$\overline{A} = \text{the smallest closed set in } X \text{ which contains } A$$
$$= \{z \in X : \text{all open neighbourhoods of } z \text{ meet } A\}$$

$$\partial A = \overline{A} \cap \overline{(X \setminus A)}$$

"A is dense in $\mathbb{R}$" means $\overline{A} = \mathbb{R}$.

$[x] = \bigcup \{A : x \in A,\ A \text{ is connected}\}$



Exercise 8.1. (a) Use the following two results,

- *a connected component of a topological space is a connected set,*
- *if the space X has a connected dense subset then X is connected,*

to show that each connected component of a topological space is a closed set.

Let C be a connected component of X.
Consider $\overline{C}$ (the closure of C in X)
$\overline{C} = \overline{C}$ meaning that C is dense in $\overline{C}$.
Also, C is connected $\Big\}\Rightarrow \overline{C}$ is connected

However, C is a maximal connected set, i.e. C cannot be contained in a strictly larger connected set $\Big\}\Rightarrow \overline{C} = C$
$C \subseteq \overline{C}$
So C is closed!

(b) Deduce from (a) that if a topological space X has finitely many connected components, then each connected component is both closed and open in X.

Assume $X = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_m$ (union of finitely many connected components). We show: C_1 is open.
$C_2, C_3, \dots, C_m$ are closed
Finite union of closed is closed $\Big\}\Rightarrow C_2 \cup C_3 \cup \dots \cup C_m$ is closed $\Rightarrow C_1 = X \setminus (C_2 \cup \dots \cup C_m)$ open!

(c) Give an example of a topological space where connected components are closed but not open. We need a space with ∞ connected components.

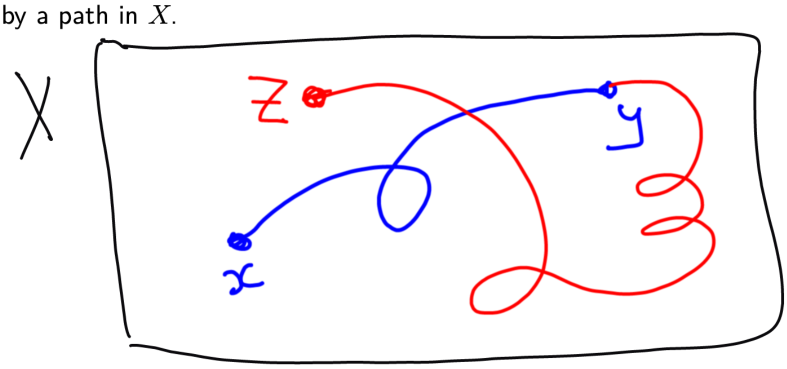
Example $\mathbb{Q}$ (viewed as a subspace of Euclidean $\mathbb{R}$)

$A \ni x$ connected $\Rightarrow A \subseteq \mathbb{R}$

A must be an interval
$A \subseteq \mathbb{Q}$ $\Big\}\Rightarrow A = \{x\}$

Therefore, connected components of $\mathbb{Q}$ are singletons.
They are closed ($\mathbb{Q} \subseteq \mathbb{R}$ is Hausdorff) but not open!

Exercise 8.2. (a) Suppose that X is a topological space, points $x, y \in X$ are joined by a path in X, and points $y, z \in X$ are also joined by a path in X. Show that x, z are joined by a path in X.



Path from x to y: cont. function $\varphi : [0,1] \to X$

$\varphi(0) = x$, $\varphi(1) = y$

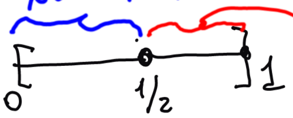
Also, $\exists \psi : [0,1] \to X$

$\psi(0) = y$, $\psi(1) = z$

We need: $\chi : [0,1] \to X$, $\chi(0) = x$, $\chi(1) = z$, χ continuous.

Idea:

$\chi(t) = \varphi(2t) \Rightarrow \chi(0) = x$, $\chi(1/2) = y$



$\chi(t) = \psi(2t-1)$

$\chi(1) = z$

(b) Furthermore, show that "$x \sim y \iff x, y$ are joined by a path in X" is an equivalence relation on X.

Equivalence classes defined by the relation $\sim$ from (b) are called **path-connected components of X.** In general, a path-connected component does not need to be open or closed in X. Nevertheless:

(c) Show that if X is an **open** subset of a **Euclidean space** $\mathbb{R}^n$, then each path-connected component of X is open. Deduce that an open connected subset of $\mathbb{R}^n$ is path-connected.