MATH31010 TOPOLOGY AND ANALYSIS

Week ~~0~~8 **lecture** B

Check In

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(To finish off Week 7:)

Reminder: Let X be a top. space, $A \subseteq X$.

$\overline{A}$ = the smallest among the closed sets in X which contain A.

Equivalently,

$\overline{A} = \{ z \in X : \text{every open nbhd of } z \text{ meets } A \}$.

New def^{n}s ① $\partial A = \overline{A} \cap \overline{(X \setminus A)}$ is the boundary of the set A (in X)

Equivalently, ∂A is the set of all points z of X such that every open nbhd of z contains a point from A and a point from $X \setminus A$.

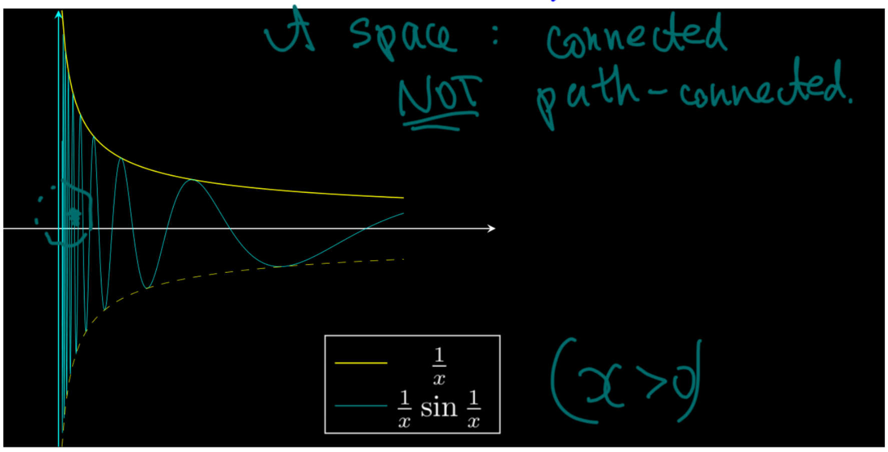
② A is dense in X if $\overline{A} = X$.

(E.g. X is dense in X)

($\mathbb{Q}$ is dense in $\mathbb{R}$)

LEMMA:

A is dense in X, A is connected $\Rightarrow X$ is connected



A space: connected

NOT path-connected.

$(x > 0)$

EXAMPLE:

$A = \{ (x, \frac{1}{x}\sin\frac{1}{x}) : x > 0 \}$

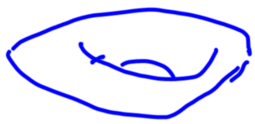
$X = A \cup \{ (0, y) : y \in \mathbb{R} \}$

$\overline{A} = X$ is connected

* Product topology $X \times Y$ $\quad \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
* Quotient topology "gluing"

↳ Examples: The Cantor space $\{0,1\}^\infty$

The torus $S^1 \times S^1$



The Möbius strip

The Klein bottle

The projective plane $\mathbb{RP}^2$

Reminder X, Y are sets $\Rightarrow$

the Cartesian product $X \times Y = \{(x,y) : x \in X, y \in Y\}$

$$X_1 \times X_2 \times \dots \times X_n = \{\overbrace{(x_1, \dots, x_n)}^{n\text{-tuples}} : x_k \in X_k \text{ for all } k = 1, \dots, n\}$$

$X_1, X_2, \dots$ a sequence of sets:

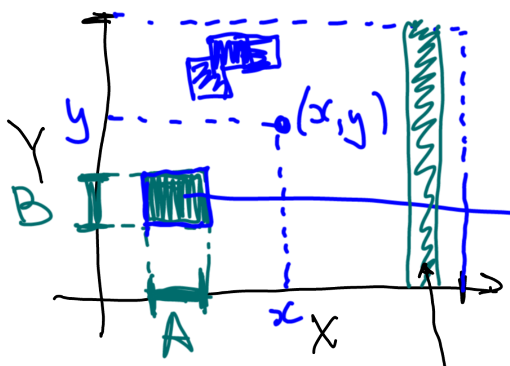
$$\prod_{k=1}^{\infty} X_k = \{(x_k)_{k \geq 1} : \forall k \in \mathbb{N}, x_k \in X_k\}$$

$\{X_\alpha : \alpha \in I\}$ a collection of sets:

$$\prod_{\alpha \in I} X_\alpha = \{(x_\alpha)_{\alpha \in I} : \forall \alpha \in I, x_\alpha \in X_\alpha\}$$

Def Let $A \subseteq X$, $B \subseteq Y$.

$A \times B$ is a subset of $X \times Y$ called *a rectangle set*.



$A \times B$ a rectangle set

a cylinder set

In particular: $A \times Y$, $X \times B$ are cylinder sets

<u>Rem</u> Intersection of two rectangle sets:

$$(A \times B) \cap (A' \times B') = \{(x,y) : x \in A, y \in B \text{ AND } x \in A', y \in B'\}$$

$$= \{(x,y) : (x \in A \ \& \ x \in A') \text{ and } (y \in B \ \& \ y \in B')\}$$

$$= (A \cap A') \times (B \cap B'),$$

is a rectangle set.

<u>Def</u> Let X, Y be top. spaces.

$$\mathcal{B} = \{ U \times V : U \underset{\text{open}}{\subseteq} X, \ V \underset{\text{open}}{\subseteq} Y \}$$

is the collection of open rectangles in $X \times Y$.

The collection $\mathcal{B}$ is the base of a topology on $X \times Y$ called the product topology.

[That is: open sets in $X \times Y$ are arbitrary unions of open rectangles.]

[<u>Rem</u> To see that $\mathcal{B}$ is a base of a topology, one needs to check:

(set from $\mathcal{B}$) $\cap$ (set from $\mathcal{B}$) = union of sets from $\mathcal{B}$

but this holds as
$$(\mathcal{U}\times V)\cap(\mathcal{U}'\times V') = (\overbrace{\mathcal{U}\cap\mathcal{U}'}^{\text{open}})\times(\overbrace{V\cap V'}^{\text{open}})$$

<u>Ex</u> Euclidean topology on $\mathbb{R}^2$ = product topology on $\mathbb{R}\times\mathbb{R}$

<u>Prop</u> The projection maps
$$p_X : X\times Y \to X, \quad (x,y)\mapsto x$$
$$p_Y : X\times Y \to Y, \quad (x,y)\mapsto y$$
are continuous.

<u>Pf</u> (for p_X) Let $\mathcal{U}\subseteq X$ be open in X.
$$p_X^{-1}(\mathcal{U}) = \{(x,y) : p_X((x,y))\in\mathcal{U}\}$$
$$= \{(x,y) : x\in\mathcal{U}\}$$
$$= \underbrace{\mathcal{U}}_{\text{open in } X}\times\underbrace{Y}_{\text{open in } Y}$$

is an open rectangle, hence open in $X\times Y$.

We have verified the def^n of "p_X is continuous". ∎