MATH31010 TOPOLOGY AND ANALYSIS   **Week 07 lecture B**

SEAtS PIN 134955

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REMINDER

COURSEWK TEST

MONDAY
13:00

Closure, Interior, Boundary & Limits

not much

DEF Let X be a top. space, $A \subseteq X$.
The **closure** of A is

$$(\mathrm{cl}(A) =) \quad \overline{A} = \bigcap \{F : A \subseteq F \subseteq X,\ F \text{ is closed in } X\}$$

The **interior** of A is

$$(\mathrm{Int}(A) =) \quad A^\circ = \bigcup \{U : U \subseteq A,\ U \text{ is open in } X\}$$

Claim $\overline{A}$ is closed in X. $\overline{A}$ is the smallest closed subset of X which contains A.
A° is open in X. A° is the largest open subset of X which is contained in A.

Proof $\overline{A}$ is closed because $\overline{A}$ is defined as an intersection of a collection of closed sets.
(Recall: intersection of any collection of closed

sets is closed.) "Smallest closed subset which contains A":

(1) $\bar{A} \supseteq A$: indeed, A is contained in all sets of the above collection, hence is contained in their intersection.

(2) $\bar{A}$ is closed (already proved)

(3) For any closed set G which contains A, $G \supseteq \bar{A}$: indeed,

$$\bar{A} = \bigcap \{F : F \text{ closed}, A \subseteq F\} =$$
$$= G \cap \bigcap \{F : F \text{ closed}, A \subseteq F, F \neq G\}$$
$$\subseteq G \text{ by def'n of } \cap.$$

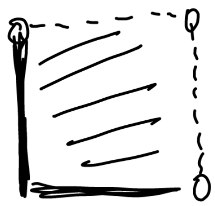
Interior: by the De Morgan Laws,

$$A^\circ = X \setminus (\overline{X \setminus A}) \leftarrow \text{use this to prove the rest}$$

[proofs for A° not given in class]

Example (from metric spaces)

$\mathbb{R}^2$

A:  $\{(x,y) : 0 \le x < 1, \; 0 \le y < 1\}$

$\bar{A}$ =  A° = 

Corollary ① A is closed in X $\iff$ $A = \bar{A}$
② A is open in X $\iff$ $A = A^\circ$

Proof of ① ($\Rightarrow$) Assume A closed. Then,

$$\bar{A} = \cancel{\bigcap \{F \text{ closed} : F \supseteq A\} =}$$
$$\cancel{= A \cap \bigcap \{F \text{ closed} : F \supseteq A, F \neq A\}}$$

Since $\underset{\text{closed}}{A} \supseteq A$, $\subseteq$ by the Claim $A \supseteq \bar{A}$.

On the other hand, $\bar{A} \supseteq A$ by the claim

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So $\overline{A} = A$.

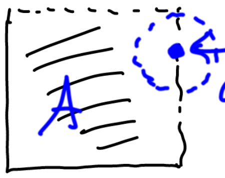
($\Leftarrow$): assume $\overline{A} = A$. Note: By the Claim, $\overline{A}$ is closed. So A is closed ■

DEF Let X be a top. space, $A \subseteq X$. A point $z \in X$ is a limit point for A if for every open nbhd $\mathcal{U}$ of z, $\mathcal{U} \cap A \neq \emptyset$.

[A point whose every open nbhd meets A is a limit point of A]

REMARK if $z \in A$ then z is a limit point of A. The converse may be false.

Ex



this is a limit point, but not a point of A.

PROP $\overline{A} = \{z \in X : z \text{ is a limit point for } A\}$

Pf We prove:

$y \notin \overline{A} \iff y$ is not a limit point for A

($\Rightarrow$) Assume $y \notin \overline{A}$ ~~$= \bigcap\{F \text{ closed} : F \supseteq A\}$. By def'n of $\bigcap$, $\exists F$ closed: $F \supseteq A$, y~~ $\notin F$.

Since $\overline{A}$ is closed, $\mathcal{U} := X \setminus \overline{A}$ is an open nbhd of y such that $\mathcal{U} \cap A = \emptyset$. So by

$[A \subseteq \overline{A} = \cancel{X} \setminus \mathcal{U}]$

def'n of a limit point, y is NOT a limit point for A.

($\Leftarrow$) Assume y is not a limit point, so

$\exists$ open $\mathcal{U} \ni y$: $\mathcal{U} \cap A = \emptyset$, equivalently $A \subseteq X \setminus \mathcal{U}$

$\mathcal{U}$ open $\Rightarrow X \setminus \mathcal{U}$ is closed $\underset{\text{claim}}{\Longrightarrow}$ $\left.\begin{array}{l} \overline{A} \subseteq X \setminus \mathcal{U} \\ y \in \mathcal{U} \end{array}\right] \Rightarrow y\ \cancel{\notin A}\ \overline{A}$. ∎

DEF If $(x_n)_{n \in \mathbb{N}}$ is a sequence in X, we say $x_n \longrightarrow x$ ("x_n converges to the point $x \in X$") if

$\forall$ open $\mathcal{U} \ni x$, $\exists N$: $x_{N+1}, x_{N+2}, \ldots \in \mathcal{U}$

FACT In non-Hausdorff spaces it is possible that $x_n \longrightarrow x$ and $x_n \longrightarrow y$ where $x \neq y$. In Hausdorff X, a limit of a sequence, if exists, is unique.

FACT It may be false that every limit point for A is a limit of a sequence from A.

THIS WILL BE TRUE IF X is Hausdorff and $\exists$ first-countable base of topology $\longleftarrow$ TRUE for metric topologies. ✓