MATH31010 TOPOLOGY AND ANALYSIS

Week 07 lecture A

852888

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Course Content

About the course

Yuri Bazlov_PreviewUser, welcome to Topology and Analysis. The tentative schedule of the course is:

I. **Introduction to Topology:** Yuri Bazlov <yuri.bazlov@manchester.ac.uk>, weeks 1-9 of semester 1.
II. **Introduction to Functional Analysis:** Yotam Smilansky <yotam.smilansky@manchester.ac.uk>, w10-12 sem1 and w1-5 sem2.
III. **Further Topics in Topology and Functional Analysis:** Donald Robertson <donald.robertson@manchester.ac.uk>, w6-11 sem2 (revision: week 12).

Coursework tests are scheduled to take place at **1pm Monday 11 November 2024 (S1 week 8)** and **11am Monday 24 February 2025 (S2 week 5).** Please check your personal timetable. The **final exam** will be in May/June 2025.

Assessment and Feedback

The assessment for the course is broken down as detailed **inside this folder. NEW:** detailed information on Coursework Test 1 is now available.

Week 05, semester 1 material

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Assessment and Feedback

ASSESSMENT - what will you receive marks for

The final mark for MATH31010 is broken down as follows:	
Coursework Test 1 in week 8 of semester 1	15%
Coursework Test 2 in week 5 of semester 2	15%
Exam in May/June	70%

More about these modes of assessment below.

Assessment: Coursework Test 1 - worth 15% of the final mark

This is a **class test** held **under exam conditions.** The test will take place at **13:00 on Monday 11 November 2024 (week 8) instead of the lecture.** The duration of the test will be **40 minutes.** We will assume the knowledge of the material up to and including the first lecture in Week 5.

It is important that you read the full specification of Coursework Test 1 here:

Assessment details

Connectedness (continued)

REMINDER "X is connected" means **NOT** $\big(\exists\, U, V$ open in X: $X = U \cup V,\ U \cap V = \emptyset,\ U \neq \emptyset,\ V \neq \emptyset\big)$

Prop For a top. space X, TFAE:

1. X is connected
2. $\forall f : X \to \mathbb{R}$ continuous, $f(X)$ is an interval; *
3. $\forall g : X \to \{0,1\}_{\text{discrete}}$ continuous, g is constant

EXAMPLE Any interval $I \subseteq \mathbb{R}$ is connected.

Def Let X be a top. space. A subset $A \subseteq X$ is a **connected set** if A is a connected space in the subspace topology.

THM $\left.\begin{array}{l} X \text{ is connected} \\ f: X \to Y \text{ continuous} \end{array}\right\} \Rightarrow f(X)$ is a connected set in Y.

In other words, a continuous image of a connected space is connected.

Pf Let $Z = f(X)$, viewed as a subspace of Y. We will show: for all $h: Z \to \mathbb{R}_{euclidean}$, $h(Z) =$ interval.

Indeed, let $h: Z \to \mathbb{R}$ be continuous. $h(Z) = h(f(X)) = (h \circ f)(X)$;

$\left.\begin{array}{l} h, f \text{ continuous} \Rightarrow h \circ f \text{ is continuous;} \\ X \text{ connected} \end{array}\right] \xRightarrow{\text{prop.}} \underbrace{(h \circ f)(X)}_{= h(Z)}$ is an interval in $\mathbb{R}$

Corollary Connectedness is a topological property.

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Connected components

Reminder An Equivalence relation on a set X:

a relation $\sim : X \times X \to \{\text{True}, \text{False}\}$

instead of "$\sim(x,y) = \text{True}$" we write $x \sim y$

False $x \not\sim y$

such that: (i) $\sim$ is reflexive: $\forall x \in X$, $x \sim x$

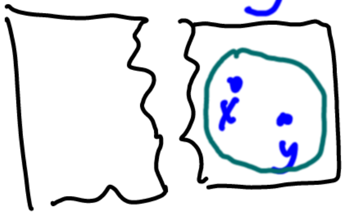
(ii) symmetric: $\forall x, y \in X$, $x \sim y \Rightarrow y \sim x$

(iii) transitive: $\forall x, y, z \in X$,

$$(x \sim y) \wedge (y \sim z) \Rightarrow x \sim z.$$

Introduce the following $\sim$ on a top. space X:

$x \sim y$ if $\exists$ connected subset $A \subseteq X$: $x, y \in A$



<u>Prop</u> $\sim$ is an equivalence rel'n on X.

<u>Proof:</u>

(i) Reflexive: need $x \sim x$. Consider $A = \{x\}$.
A is connected (exercise) so $x \sim x$ by def'n of $\sim$.

(ii) Symmetric: let $x \sim y$. Then $\exists$ connected $A \subseteq X$: $x, y \in A$. Then $y, x \in A \Rightarrow y \sim x$.

(iii) <u>LEMMA</u> If $A, B \subseteq X$ are connected sets, $A \cap B \neq \emptyset$, then $A \cup B$ is a connected set.

Pf of Lemma: Suppose $g: A \cup B \to \{0,1\}$ discrete is a continuous function. Also, $A \cap B \neq \emptyset$, let $y \in A \cap B$.

$g|_A : A \to \{0,1\}$ (Connected) $\quad (g|_A = g \circ in_{A \to A \cup B})$ is continuous

$\Rightarrow g|_A$ is constant: $g(A) = \{g(y)\}$

Also, $g|_B : B \to \{0,1\}$ (connected) continuous $\Rightarrow g(B) = \text{const} = \{g(y)\}$

Therefore, $g(A \cup B) = \{g(y)\}$ and so g is constant on $A \cup B \underset{\text{prop.}}{\Longrightarrow} A \cup B$ is connected. $\blacksquare$

(ii) Transitive: assume $(x \sim y) \wedge (y \sim z)$, so $\exists$ A connected: $x, y \in A$
$\exists$ B connected: $y, z \in B$

Then $x, z \in A \cup B$, $A \cap B \ni y \Rightarrow$ by LEMMA A, $A \cup B$ connected $\Rightarrow x \sim z$. $\blacksquare$

FACT (about partitions). In set theory, a **partition** of a set X is a collection $\mathcal{P} = \{ P_\alpha : \alpha \in I \}$ of subsets of X such that

* $\forall \alpha \in I$, $P_\alpha \neq \emptyset$;
* $\forall \alpha, \beta \in I$, $\alpha \neq \beta$, $P_\alpha \cap P_\beta = \emptyset$;
* $X = \bigcup_{\alpha \in I} P_\alpha$.

If $\sim$ is an equivalence relation on X, for each $x \in X$ define $[x] = \{ y \in X : x \sim y \} \subseteq X$ the equivalence class of x.

The collection of all equivalence classes is a partition of X.

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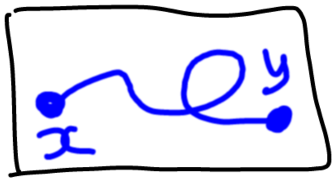
DEF Let X be a topological space, $\sim$ be the equivalence rel'n considered above
($x \sim y \iff \exists\, A$ connected, $A \subseteq X$, $x, y \in A$)
Connected components of X are the equivalence classes on X with respect to $\sim$.

Prop If $h: X \to Y$ is a homoeomorphism, then h maps connected components of X to connected components of Y.

Def Let X be a top. space. A **path** in X is a continuous function $\varphi : [0,1] \to X$.

X



Points x, y of X are **connected by a path** if $\exists$ path $\varphi : [0,1] \to X$ such that $\varphi(0) = x$, $\varphi(1) = y$.

Def A top. space X is **path-connected** if $\forall x, y \in X$, x and y are connected by a path.

Prop X is path-connected $\Rightarrow$ X is connected.

Exercise

① $\mathbb{R}^2 \setminus \{(0,0)\}$ is path-connected.
punctured plane

② $\mathbb{R} \setminus \{0\}$ is disconnected.

③ $\mathbb{R}$ is not homeo to $\mathbb{R}^2$.