700848

Topology Feedback Quiz, week 5: Hausdorffness and compactness

Open books. 10–15 minutes. Not for credit. To be marked in class.

Felix Hausdorff (1868–1942)
a founder of Topology

Question 1 ♣ The closed bounded interval $[10, 20]$ in $\mathbb{R}$ is:

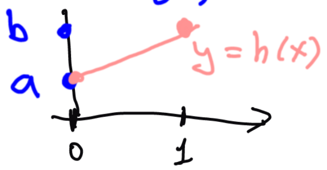
Recall: Metric spaces (e.g. Reuclidean) are Hausdorff

✓ Hausdorff compact ○ Hausd. non-comp. ~~○ non-Haus. comp.~~ ~~○ non-H. non-comp.~~

The Heine-Borel Lemma says: $[0,1]$ is compact.

$\forall [a,b]$ (where $a<b$), $\exists$ homeomorphism $h:[0,1]\to[a,b]$:

e.g. $h(x) = a + (b-a)x$



The half-open interval $[0, 1)$ in $\mathbb{R}$ is

○ Hausdorff compact ✓ Hausd. non-comp. ~~○ non-Haus. comp.~~ ~~○ non-H. non-comp.~~

Recall: a compact set in a Hausdorff space is closed.
$[0,1)$ is not closed in $\mathbb{R} \Rightarrow$ not compact.

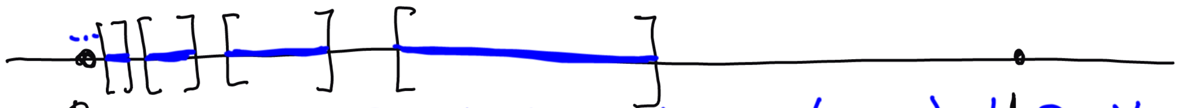
The union $[0, 1/3] \cup [2/3, 1]$ is

✓ Hausdorff compact ○ Hausd. non-comp. ~~○ non-Haus. comp.~~ ~~○ non-H. non-comp.~~

Recall: a closed subset of a compact is compact.
$[0,1/3]\cup[2/3,1]$ is closed in $\underbrace{[0,1]}_{\text{compact by Heine-Borel}}$ (union of 2 closed sets)

The set $\bigcup_{n=1}^{\infty}[\frac{1}{2n+1}, \frac{1}{2n}]$ ~~is~~ $= X$

○ Hausdorff compact ✓ Hausd. non-comp. ~~○ non-Haus. comp.~~ ~~○ non-H. non-comp.~~



I claim that $0 \in \mathbb{R}\setminus X$ but $\forall \varepsilon>0$, $(-\varepsilon,\varepsilon) \not\subset \mathbb{R}\setminus X$

Indeed, if $\varepsilon>0$, take n: $\frac{1}{2n} < \varepsilon \Rightarrow (-\varepsilon,\varepsilon)\cap X \ni \frac{1}{2n}$

So $\mathbb{R}\setminus X$ is not open $\Rightarrow X$ is not closed

The set $\{0\} \cup \bigcup_{n=1}^{\infty}[\frac{1}{2n+1}, \frac{1}{2n}]$ is
Closed subset of $[0,1]$

✓ Hausdorff compact ○ Hausd. non-comp. ~~○ non-Haus. comp.~~ ~~○ non-H. non-comp.~~

$\mathbb{R}$ with cofinite topology is

Intersection of two cofinite sets is a cofinite set $\Rightarrow \neq \emptyset$

○ Hausdorff compact ○ Hausd. non-comp. ✓ non-Haus. comp. ○ non-H. non-comp.

$\mathbb{R}$ with antidiscrete topology is

○ Hausdorff compact ○ Hausd. non-comp. ○ non-Haus. comp. ○ non-H. non-comp.

Cofinite topology is compact:

Suppose $X = \bigcup \mathcal{C}$ where $\mathcal{C}$ is an open cover of $(X, \text{cofinite})$. Assume $X \neq \emptyset$ (otherwise $\emptyset$ is compact) and take $U \in \mathcal{C}$ such that $U \neq \emptyset$.

Then $U = X \setminus F$, F is finite.

$= X \setminus \{x_1, x_2, \dots, x_n\}$

x_1 must be covered by some $V_1 \in \mathcal{C}$ (because $\mathcal{C}$ covers the whole of X)

x_2 ——— " ——— $V_2 \in \mathcal{C}$

.... x_n covered by $V_n \in \mathcal{C}$

Then $\{U, V_1, V_2, \dots, V_n\}$ is an open finite subcover of $\mathcal{C}$. So cofinite $\Rightarrow$ compact.

Question 2 ♣ Suppose T_{weak}, T_{strong} are topologies on a set X such that T_{strong} is stronger than T_{weak}. What must be true?

- [x] the function $id_X : (X, T_{strong}) \to (X, T_{weak})$ is continuous
- [ ] the function $id_X : (X, T_{strong}) \to (X, T_{weak})$ is a homeomorphism X
- [x] T_{weak} is Hausdorff implies T_{strong} is Hausdorff ← shown in lectures
- [x] T_{strong} is compact implies T_{weak} is compact ← exercise

Antidiscrete is weaker than Cofinite Compact

$\Longrightarrow$ by Q2.4 antidiscrete is compact.