COURSEWORK:

- the first COURSEWORK ASSESSMENT TEST will take place on MONDAY 11 NOVEMBER, 13:00 (the test will replace the lecture)
  worth 15% of the final mark for MATH31010
- the material covered in the lecture today is <u>NOT</u> INCLUDED in the test;
- the material covered in the TUTORIAL may be useful for the test.

Connectedness

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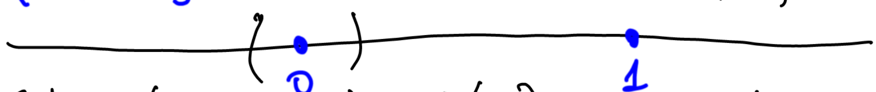
DEF A top. space X is **disconnected** if $\exists\, U, V_{open} \subseteq X$:
$U \cap V = \emptyset$, $U \neq \emptyset$, $V \neq \emptyset$, $X = U \cup V$.
[A disconnected space is a disjoint union of two non-empty open sets.]
X is **connected** if X is not disconnected.

EX ⊛ $X = \underbrace{(0,1)}_{U} \cup \underbrace{(2,3)}_{V} \subseteq \mathbb{R}_{euclidean}$ is disconnected.
U, V are open in X
$U \cap V = \emptyset$, $X = U \cup V$

⊛ $X = \{0, 1\} \subseteq \mathbb{R}_{euclidean}$: $U = \{0\}$, $V = \{1\}$



$\left.\begin{array}{l} U = (-\infty, 1) \cap X \\ V = (0, +\infty) \cap X \end{array}\right\}$ so U, V are open in X

In fact, $\{0,1\}$, as a subspace of $\mathbb{R}$, has discrete topology.

Prop For a top. space X TFAE:

1. X is connected
2. $\forall f: X \to \mathbb{R}$ continuous, $f(X)$ is an interval;
3. $\forall g: X \to \{0,1\}_{\text{discrete}}$ continuous, g is constant

Def An *interval* is a subset I of $\mathbb{R}$ such that $\forall\, x,y \in I$, $\forall\, t: x<t<y$, $t \in I$.

FACT (can be proved by considering $\sup I$ and $\inf I$)
$I \subseteq \mathbb{R}$ is an interval $\Leftrightarrow$
I is (a,b), $[a,b)$, $(a,b]$, $[a,b]$, $(a,+\infty)$, $[a,+\infty)$, $(-\infty,b)$, $(-\infty,b]$, or $\mathbb{R}$.

Pf of the Proposition.
(i) $\Rightarrow$ (ii): I will prove this by contrapositive.
Assume $f: X \to \mathbb{R}$ continuous, $f(X)$ is not an interval $\leftarrow$ so $\exists\, a,b \in f(X)$; $\exists\, t: a<t<b$, $t \notin f(X)$.

Put $\mathcal{U} = f^{-1}((-\infty,t)) \supseteq f^{-1}(\{a\}) \neq \emptyset$
$V = f^{-1}((t,+\infty)) \supseteq f^{-1}(\{b\}) \neq \emptyset$

these are open because $(-\infty,t)$, $(t,+\infty)$ are open in $\mathbb{R}$ and f is continuous.

$\mathcal{U} \cap V = f^{-1}((-\infty,t)\cap(t,+\infty)) = f^{-1}(\emptyset) = \emptyset$
so $\mathcal{U}, V$ are disjoint in X.

$$\mathcal{U} \cup V = f^{-1}(\mathbb{R} \setminus \{t\}) = X \setminus f^{-1}(\{t\})$$
$$= X \text{ as } t \notin f(X)$$

and so, by def'n, X is disconnected.

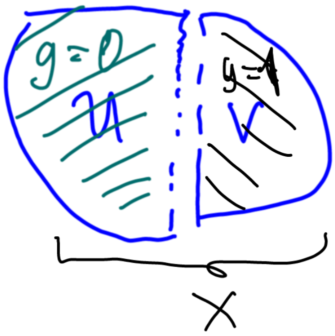
(ii) $\Rightarrow$ (iii) : Let $g : X \to \{0,1\}_{\text{discrete}}$ be continuous.

$in_{\{0,1\}} : \{0,1\} \hookrightarrow \mathbb{R}$ continuous

So $in_{\{0,1\}} \circ g : X \to \mathbb{R}$ is continuous

by (ii), the image of X in $\mathbb{R}$ must be an interval. Yet it is a subset of $\{0,1\}$ so it is at most one point $\Rightarrow$ g is constant

(iii) $\Rightarrow$ (i) : by contrapositive. Assume X disconnected,
$X = \mathcal{U} \cup V$, $\mathcal{U} \cap V = \emptyset$, $\mathcal{U}, V$ open in X, $\mathcal{U} \neq \emptyset$, $V \neq \emptyset$



Define $g : X \to \{0,1\}$ discrete,

$$g(x) = \begin{cases} 0, & x \in \mathcal{U} \\ 1, & x \in V \end{cases}$$

g is continuous: $g^{-1}(\emptyset) = \emptyset$

$g^{-1}(\{0\}) = \mathcal{U}$ $\quad g^{-1}(\{1\}) = V$ $\quad g^{-1}(\{0,1\}) = X$

the preimage of every open subset of $\{0,1\}$ is open in X. g is not constant because

$g(\mathcal{U}) = \{0\}$ $\quad \mathcal{U} \neq \emptyset$
$g(V) = \{1\}$ $\quad V \neq \emptyset$ $\quad \boxed{\Xi}$

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EXAMPLE Every interval $I \subseteq \mathbb{R}$ is a connected space.

Explanation: use (ii).

Let $f: I \to \mathbb{R}$ be continuous.

By the Intermediate Value THM (see MFA year 1) $f(I)$ is an interval.

So by PROP., I is <u>connected</u>.