MATH31010 TOPOLOGY AND ANALYSIS **Week 05 lecture A**

SEAtS 706945

Reminder We proved:

* a closed subset of a compact is compact
* a continuous image of a compact is compact
* in a Hausdorff space, a compact set is closed.

Coursework Test: expect it to be moved from 7^{th} Nov to a lecture / tutorial in Week 7 (Fri) / week 8 (Mon)

THM (the Topological Inverse Function THM, **TIFT**)

If K is a compact space and Y is Hausdorff, and $f : K \to Y$ is a continuous bijection, then f is a homeomorphism.

Pf Assumptions $\Rightarrow$ the only thing needing a proof is:

f^{-1} is continuous. $f^{-1}: Y \to K$

Use the closed set criterion of continuity:

f^{-1} is continuous $\iff$ $\forall \underset{\text{closed}}{F \subseteq K}$, $(f^{-1})^{-1}(F)$ is closed in Y.

Assume $F \subseteq K$ closed.

Then F is compact.

Then $(f^{-1})^{-1}(F) = \underbrace{f}_{\text{continuous}}(F)$ is compact. ⊗

$\underset{\text{Hausdorff}}{Y} \supseteq \underset{\text{compact}}{f(F)} \Rightarrow f(F)$ is closed. ⊗

We have verified the ⊗ closed set criterion of continuity $\Rightarrow$ f^{-1} is continuous. ∎

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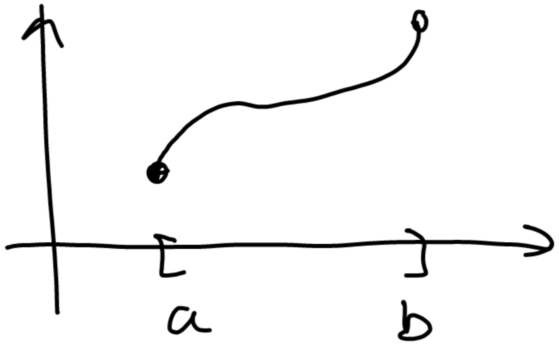
Rem We will soon see that this implies IFT from Real Analysis:



$f : [a,b] \to \mathbb{R}$
monotone
continuous

$\Rightarrow \exists\, g = f^{-1} : [c,d] \to [a,b]$ continuous

Ex X is a finite set $\Longrightarrow$ every topology on X is compact (Exercise]

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Compactness in metric and Euclidean spaces

Prop A compact subset of a metric space is closed and bounded.

[Reminder: K is "bounded" $\iff \exists r>0, \exists x \in X$: $K \subseteq B_r(x)$]

Pf Let X be a metric space. Assume $K \subseteq X$ is compact.

$\left.\begin{array}{l} X \text{ is Hausdorff} \\ \text{In Hausdorff, compacts are closed} \end{array}\right\} \Rightarrow$

$\Rightarrow K$ is closed in X. Fix a point $x \in X$ and consider $\{B_r(x)\}_{r \in \mathbb{R}_{>0}} = \mathcal{C}$. Clearly $\bigcup_{r \in \mathbb{R}_{>0}} B_r(x) = X$.

K is compact, so by Criterion of compactness for subsets, a finite subcollection of $\mathcal{C}$, say $\{B_{r_1}(x), B_{r_2}(x), \dots, B_{r_n}(x)\}$, covers K.

$$K \subseteq B_{r_1}(x) \cup \dots \cup B_{r_n}(x) = \underbrace{B_{\max(r_1, \dots, r_n)}(x)}_{\text{this shows } K \text{ is bounded.}} \quad \blacksquare$$

THM (the Heine-Borel Lemma)
The closed bounded interval $[0,1]$ is a compact subset of $\mathbb{R}$ (Euclidean line).

Pf (proof by halving the interval) Assume for contradiction that $[0,1]$ has some cover $\mathcal{C}$ by open subsets of $\mathbb{R}$ such that no finite subcollection of $\mathcal{C}$ covers $[0,1]$.

At least one of the

$$\left[0, \tfrac{1}{2}\right] \qquad\qquad \left[\tfrac{1}{2}, 1\right]$$

does not have a finite subcover in $\mathcal{C}$

(because: if $[0, 1/2]$ has finite cover $\mathcal{C}_1 \subseteq \mathcal{C}$,

$[1/2, 1]$ has finite cover $\mathcal{C}_2 \subseteq \mathcal{C}$,

then $[0,1]$ has cover $\mathcal{C}_1 \cup \mathcal{C}_2$ which is finite)

Denote by $[a_1, b_1]$ the half of $[0,1]$ which has no finite subcover in $\mathcal{C}$.

$$[a_1, b_1] \qquad 0 \le a_1 \le b_1 \le 1, \qquad b_1 - a_1 = \frac{1}{2}$$

In the same way, choose $[a_2, b_2] \subseteq [a_1, b_1]$ /one of the halves/

$$a_1 \le a_2 \le b_2 \le b_1, \quad b_2 - a_2 = \frac{1}{2^2},$$

$[a_2, b_2]$ has no finite subcover in $\mathcal{C}$.

$n \in \mathbb{N}$: $\quad [a_n, b_n] \qquad a_{n-1} \le a_n, \quad b_n - a_n = \frac{1}{2^n}$

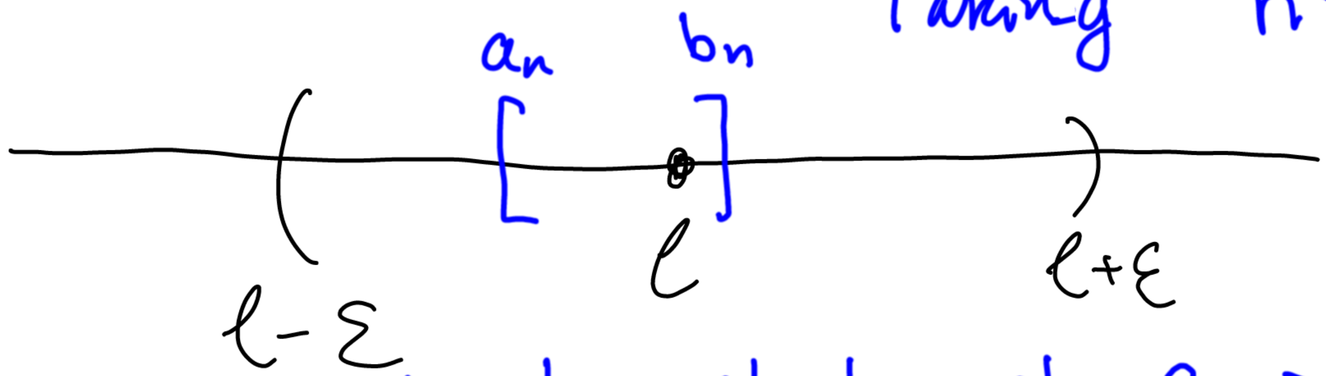
has no finite subcover in $\mathcal{C}$.

The sequence $(a_n)_{n\geq 1}$ is increasing and bounded $\Rightarrow \exists\, \ell = \lim_{n\to\infty} a_n$. Note: $\ell \in [0,1]$

$\Rightarrow \exists\, \mathcal{U} \in \mathcal{C} : \ell \in \mathcal{U}$. $\mathcal{U}$ open $\Rightarrow$

$\Rightarrow \exists\, \varepsilon > 0 : \quad B_\varepsilon(\ell) = (\ell - \varepsilon, \ell + \varepsilon) \subseteq \mathcal{U}$.

Taking $n : \frac{1}{2^n} < \varepsilon$,



we observe that $|a_n - \ell|, |b_n - \ell| < \varepsilon \Rightarrow$ $[a_n, b_n]$ is covered by ONE SET $\mathcal{U} \in \mathcal{C}$.