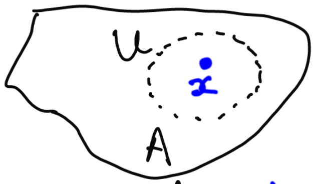
**Week 4 tutorial**

**Exercise 4.1** (basic test of openness)**.** Suppose that $\mathcal{B}$ is a base of a topology on $X$, and call the subsets of $X$ which are members of $\mathcal{B}$ **basic open sets.**

Let $A$ be a subset of $X$. Prove that the following are equivalent:

1. $A$ is open in $X$.
2. $A$ is a union of a collection of basic open sets.
3. For each point $x \in A$, there exists a basic open set $U$ such that $x \in U$ and $U \subseteq A$.

3:




$x \in U$ & $U \subseteq A$

1 $\Leftrightarrow$ 2 by definition of base.

2 $\Rightarrow$ 3 given $A = \bigcup_{\alpha \in I} U_\alpha$, $\forall \alpha, U_\alpha \in \mathcal{B}$

Take any $x \in A$. Then $x \in \bigcup_{\alpha \in I} U_\alpha$, and so by definition of union, $\exists \beta \in I$: $x \in U_\beta$.

$x \in U_\beta \subseteq \bigcup_{\alpha \in I} U_\alpha = A$ so 3 holds.

3 $\Rightarrow$ 2: given: $\forall x \in A\ \exists U_x \in \mathcal{B}$ : $x \in U_x \subseteq A$

I claim that $A = \bigcup_{x \in A} U_x$. Indeed,


$\forall x' \in A$, $x' \in U_{x'} \subseteq \bigcup_{x \in A} U_x$

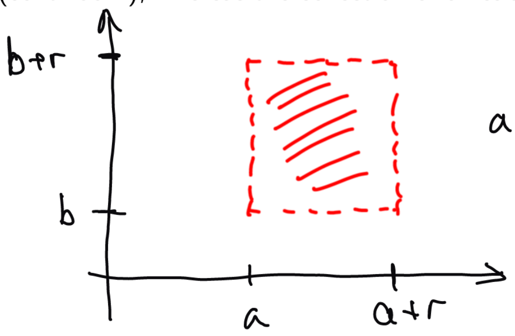
and so $A \subseteq \bigcup_{x \in A} U_x$.

On the other hand, $\forall x\ U_x \subseteq A$ and so

$\bigcup_{x \in A} U_x \subseteq A$. So $A = \bigcup_{x \in A} U_x$.

**Exercise 4.2** (the Euclidean topology has a countable base)**.** Consider the Euclidean space $\mathbb{R}^2$, and let $\mathcal{Q}$ be the (countable) collection of all open squares in $\mathbb{R}^2$ where the coordinates of all four vertices are rational numbers. Prove that $\mathcal{Q}$ is a base for the Euclidean topology.

Deduce that the collection of all open sets in the Euclidean space $\mathbb{R}^2$ has cardinality $\aleph$ (continuum), whereas the collection of all subsets of $\mathbb{R}^2$ has cardinality $2^{\aleph}$.

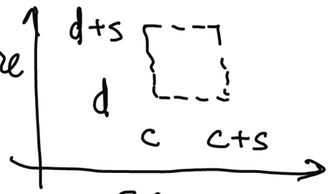


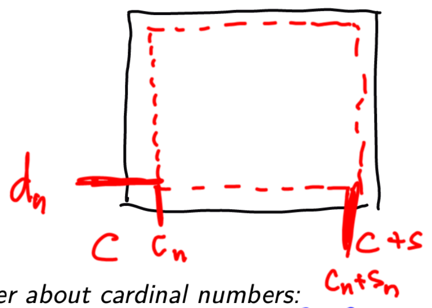

$a, b, r \in \mathbb{Q}$

$\{(x,y) \in \mathbb{R}^2 : a < x < a+r,\ b < y < b+r\}$

All such open squares are open sets in Euclidean topology.

1. Show that every open square (not necessarily rational) is a union of rational open squares.




Let $(c_n)$ be a sequence of rational numbers, $\lim_{n\to\infty} c_n = c$, $c_n \geq c$. Similarly $s_n \nearrow s$, $d_n \searrow d$ (and $c_n+s_n \leq c+s$)($d_n+s_n \leq d+s$)

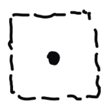
*Reminder about cardinal numbers:*

Then $\bigcup_{n \geq n} Q^{s_n \times s_n}_{(c_n, d_n)} = Q^{s \times s}_{(c,d)}$

2. Show that every open set in $\mathbb{R}^2$ is a union of open squares in $\mathbb{R}^2$.

Consider the metric on $\mathbb{R}^2$ known as $d_\infty$:

$d_\infty((x_1,y_1),(x_2,y_2)) = \max(|x_1-x_2|, |y_1-y_2|)$



Open balls = open squares

So every $d_\infty$-open set is a union of open squares. $d_\infty$ is Lipschitz-equivalent to $d_2$, so $d_\infty$-topology = $d_2$-topology (Euclidean)

$\boxed{\#\{\text{open subsets of } \mathbb{R}^2\} = \aleph}$

$\# 2^{\mathbb{R}^2} = 2^{\aleph} > \aleph$

- $\aleph_0$ *(aleph-zero) denotes the countably infinite cardinality, e.g., the cardinality of* $\mathbb{N}$;
- $\aleph$ *(aleph) denotes the cardinality of continuum, e.g., the cardinality of* $\mathbb{R}$,
- *one has* $|\mathbb{R}| = \aleph = 2^{\aleph_0} = |P(\mathbb{N})| > \aleph_0$.

**Exercise 4.3** (subbase)**.** Let $(Y, \mathcal{T})$ be a topological space. A **subbase** of $\mathcal{T}$ is a collection $\mathcal{S}$ of open sets such that **finite intersections of sets from** $\mathcal{S}$ **form a base of** $\mathcal{T}$**.**

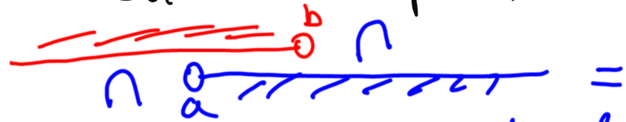
It is worth noting that, given any set $Y$ (without topology) and any collection $\mathcal{S}$ of subsets of $Y$, we can construct a topology $\mathcal{T}_{\mathcal{S}}$ on $X$ by using $\mathcal{S}$ as a subbase. That is, $\mathcal{T}_{\mathcal{S}}$ consists of arbitrary unions of finite intersections of members of $\mathcal{S}$. It is not difficult to show that this collection $\mathcal{T}_{\mathcal{S}}$ is a topology.

Prove that the collection of all **open rays** in the real line, i.e., sets of the form $(-\infty, a)$ and $(b, +\infty)$, is a subbase of the Euclidean topology.

$\mathcal{S}$ is a subbase $\iff \{ U_1 \cap \dots \cap U_n \mid U_1, \dots, U_n \in \mathcal{S} \}$ is a base of the topology

$\{ (-\infty, a)$, $(b, +\infty) \}$ is a subbase of Euclidean topology.

These sets are Euclidean-open.

Also:  $= (a,b)$ open interval.

Open intervals form a base of the Euclidean topology.

**Exercise 4.4** (subbasic test of continuity)**.** Let $X$, $Y$ be topological spaces, $f\colon X \to Y$ be a function, and $\mathcal{S}$ be a subbase of topology on $Y$. Prove that the following are equivalent:

1. $f$ is continuous.
2. The preimage of every subbasic subset of $Y$ is open in $X$.

(Note that 2. means $\forall V \in \mathcal{S}$, $f^{-1}(V)$ is open in $X$)

**Exercise 4.5.** (a) Let $X$ be a topological space and let $f\colon X \to \mathbb{R}$ be a function. Prove: $f$ is continuous iff for all $a, b \in \mathbb{R}$, the sets $X_{f<a} = \{x \in X : f(x) < a\}$ and $X_{f>b} = \{x \in X : f(x) > b\}$ are open in $X$.

(b) Let $X$ be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous functions. Prove that the function $f + g\colon X \to \mathbb{R}$ is continuous. Hint: use (a).