MATH31010 TOPOLOGY AND ANALYSIS   **Week 04 lecture B**

SEAtS 210833

THM A continuous image of a compact is compact.

TERMINOLOGY: "a compact" = a compact topology

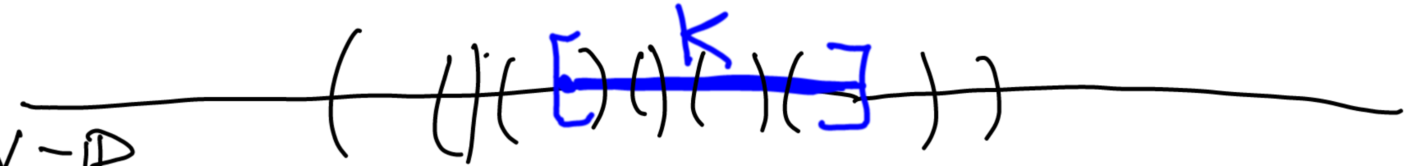
"A compact set" or "a compact subset of X" means a subset $K \subseteq X$ such that K, with subspace topology, is compact.

LEMMA (criterion of compactness for a subset) Let $K \subseteq X$.

The following are equivalent: ① K is a compact set.

② Any collection $\mathcal{F}$ of open sets in X

which covers K (that is, $K \subseteq \bigcup \mathcal{F}$), has a finite subcollection which still covers K.



$X = \mathbb{R}$

(Pf in the notes - not given in class)

PROOF That a continuous image of a compact is compact. $f: X \to Y$ is continuous, X is compact $\Longrightarrow$ $f(X)$ is a compact subset of Y.

Use LEMMA: let $\mathcal{G}$ be a collection of open sets in Y which covers $f(X)$.

Consider $\mathcal{C} = \{ \underbrace{f^{-1}(V)}_{\text{open sets in } X \text{ because } f \text{ is continuous}} : V \in \mathcal{G} \}$

I claim that $\mathcal{C}$ is an <u>open</u> cover of X.

$$\bigcup \mathcal{C} = \bigcup \{ f^{-1}(V) : V \in \mathcal{G} \} = f^{-1}\Big(\bigcup \mathcal{G} \Big)$$

$$\supseteq f^{-1}\big(\underline{f(X)} \big) \quad \text{because } \bigcup \mathcal{G} \supseteq f(X)$$

$$\parallel$$

$$\{ x \in X : f(x) \in f(X) \} = X$$

Since X is compact, its open cover $\mathcal{C}$ has a finite subcover, say $f^{-1}(V_1), \ldots, f^{-1}(V_n)$ where $V_1, \ldots, V_n \in \mathcal{C}$. Then

$$X = f^{-1}(V_1) \cup \cdots \cup f^{-1}(V_n)$$

$$\Rightarrow \quad f(X) = f\left[f^{-1}(V_1) \cup \cdots \cup f(V_n)\right]$$

[$f(f^{-1}(V_1)) = V_1$ is wrong but $f(f^{-1}(V_1)) \subseteq V_1$]

$$\subseteq V_1 \cup \cdots \cup V_n.$$

We have verified Criterion of Compactness for $f(X)$. $\blacksquare$

COROLLARY Compactness is a topological property. [**Proof:** $\underset{\text{compact}}{X} \xrightarrow[f]{\cong} Y$: $Y = f(X)$ compact by THM]

Prop A closed subset of a compact is compact.

Proof X = compact top. space

F = closed set in X. Let $\mathcal{C}$ be a collection of open sets in X such that $F \subseteq \bigcup \mathcal{C}$.

$\mathcal{C} \cup \{\underbrace{X \setminus F}_{\text{open}}\}$ is open cover of X

$\Rightarrow \exists$ finite subcover $U_1, U_2, \dots, U_n, \overbrace{X \setminus F}^{\text{possibly}}$

If this covers X then $U_1, \dots, U_n$ must cover F.
We have verified the Criterion $\Rightarrow$ F is compact.

PROP In a Hausdorff space, a compact set is closed.

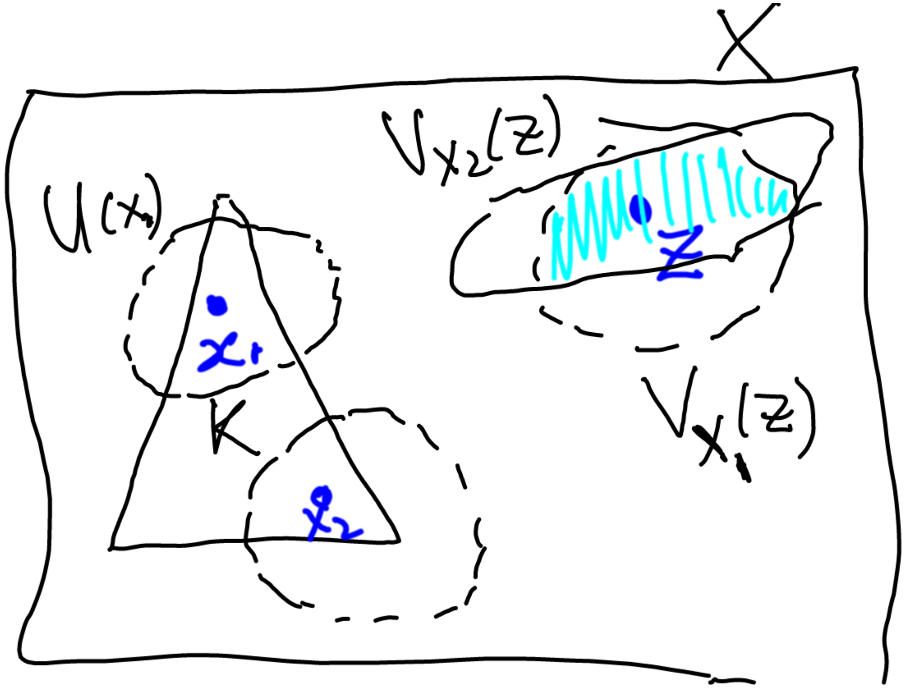
Proof We will show: (X = Hausdorff, $\underset{\text{compact}}{K} \subseteq X$)

$\forall z \in X \setminus K$, $\exists\, V(z)$ open, $z \in V(z) \subseteq X \setminus K$

This is enough to show K closed:

$$X \setminus K = \bigcup_{z \in X \setminus K} V(z)$$

union of open sets.



$\forall x \in K, \quad x \neq z \implies$ Hausdorff

$\exists$ open $U(x) \ni x,\ V_x(z) \ni z$

$U(x) \cap V_x(z) = \emptyset$

K is covered by $\{U(x) : x \in K\}$

So by Criterion, $\exists\ x_1, \ldots, x_n \in K$:

$$K \subseteq U(x_1) \cup U(x_2) \cdots \cup U(x_n)$$

Take $z \in V(z) = V_{x_1}(z) \cap V_{x_2}(z) \cap \cdots \cap V_{x_n}(z)$

open (finite intersection)

$$V(z) \cap \left(\mathcal{U}(x_1) \cup \cdots \cup \mathcal{U}(x_n) \right)$$

$$= \bigcup_{i=1}^{n} V(z) \cap \mathcal{U}(x_i) = \emptyset \quad \text{so} \quad V(z) \cap K = \emptyset.$$

$\blacksquare$