SEAtS PIN 236145

Example (of a non-Hausdorff space)
$X = \{1, 2\}$ with the antidiscrete topology is NOT Hausdorff. The only open set which contains 1 is $\{1,2\}$ and so it is for 2. Hence 1 and 2 have no disjoint open nbhds.

Def (stronger topology, weaker topology)
Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on the same set X. We say that the topology $\mathcal{T}'$ is stronger than $\mathcal{T}$ if every set open in $\mathcal{T}$ is also open in $\mathcal{T}'$.

stronger means more open sets

In other words, $\mathcal{T} \subseteq \mathcal{T}'$. In this case, $\mathcal{T}$ is weaker than $\mathcal{T}'$.

Prop If $(X, \mathcal{T})$ is Hausdorff and $\mathcal{T}'$ is **stronger** than $\mathcal{T}$, then $(X, \mathcal{T}')$ is Hausdorff.

Pf To prove that $(X, \mathcal{T}')$ is Hausdorff, take any two points $x, y \in X$, $x \neq y$. Using the assumption that $\mathcal{T}$ is Hausdorff, find the open nbhds $U \ni x$, $V \ni y$, $U \cap V = \emptyset$, U, V open in topology $\mathcal{T}$. $[U, V \in \mathcal{T}]$

Since $\mathcal{T}'$ is stronger than $\mathcal{T}$, U, V are open in $\mathcal{T}'$ and so we have verified the def. of Hausdorff for $\mathcal{T}'$. ∎

Q What is the weakest topology on a given set X? A. $(X, \text{antidiscrete}) = (X, \{\emptyset, X\})$

Strongest? $(X, \text{discrete}) = (X, \mathcal{P}(X)) = (X, 2^X)$
↑ notation

Is $(X, \text{discrete})$ always Hausdorff?
Yes. Can verify the def'n of Hausdorff directly:
$x, y \in X$, $x \neq y$, put $U = \{x\}$, $V = \{y\}$, open.
ALSO Can introduce metric: $d^*(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$
(the discrete metric on X)
τ_{d^*} = the discrete topology.

Must (X, antidiscrete) be non-Hausdorff?

No: if $X = \emptyset$ or $|X| = 1$, then X is Hausdorff

Rem Non-Hausdorff spaces are standard in algebraic geometry [the Zariski topology]

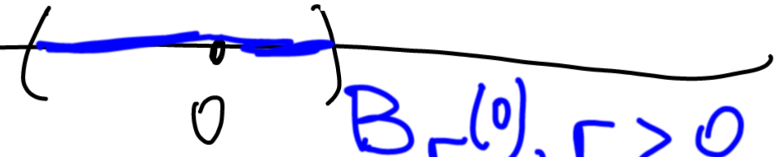
Compactness

Reminder An open cover of a top. space X is a collection $\{U_\alpha : \alpha \in I\}$ where $U_\alpha \underset{\text{open}}{\subseteq} X$ $(\forall \alpha)$ and $\bigcup_{\alpha \in I} U_\alpha = X$.

A **subcover** of an open cover $\mathcal{F}$ of X is a subcollection of $\mathcal{F}$ which is an open cover of X.

DEF A top. space X is **compact** if every open cover of X has a **FINITE** subcover.

Non-example. $(\mathbb{R}, \text{Euclidean})$



$B_r(0), r > 0$

$\{ B_r(0) : r \in \mathbb{R}_{>0} \}$ is an open cover of $\mathbb{R}$. $\forall r > 0$, $B_r(0)$ is open; $\mathbb{R} = \bigcup_{r>0} B_r(0)$.

THIS OPEN COVER OF $\mathbb{R}$ HAS NO FINITE subcovers

So $(\mathbb{R}, \text{Euclidean})$ is NOT compact.

THM A continuous image of a compact is compact.

[**Rem** "a compact" = a compact ~~topological~~ space ; "compact" is also an adjective.

Comment Meaning: Let X be a compact top. space. Let $f : X \to Y$ be a continuous function. Then $f(X)$, considered as a subspace of the top. space Y, is itself a compact topological space.