MATH31010 TOPOLOGY AND ANALYSIS **Week** 03 **lecture** B

Reminder: Hausdorff spaces SEAtS PIN 371477

Def A topological space X is **Hausdorff** if

$$\forall x, y \in X, \quad x \neq y \Rightarrow \exists\, U, V \text{ open in } X: \; x \in U,\; y \in V,\; U \cap V = \emptyset$$

I.e. two distinct points of X must have disjoint open neighbourhoods.

PROP: the Hausdorff property is a topological property.

(I.e. if X is Hausdorff and $Y \xrightarrow{\sim} X$ then Y is also Hausdorff)

Pf Assume $Y \underset{f}{\xrightarrow{\sim}} X$, X Hausdorff. Need to show: Y Hausdorff, which means:

$\forall a, b \in Y,\ a \neq b:\quad \exists\, \mathcal{U}_a, \mathcal{U}_b \underset{\text{open}}{\subseteq} Y:\quad a \in \mathcal{U}_a,\ b \in \mathcal{U}_b,\ \mathcal{U}_a \cap \mathcal{U}_b = \emptyset$

Construction of $\mathcal{U}_a, \mathcal{U}_b$: consider $f(a), f(b) \in X$.
$f(a) \neq f(b)$ because f is injective.
X is Hausdorff so $\exists\ V_a \ni f(a),\ V_b \ni f(b),\ V_a \cap V_b = \emptyset$, V_a, V_b open in X.

Put $\mathcal{U}_a = f^{-1}(V_a)$, $\mathcal{U}_b = f^{-1}(V_b)$: $\mathcal{U}_a, \mathcal{U}_b \subseteq Y$

(1) $\mathcal{U}_a \ni a = f^{-1}(f(a))$; $\mathcal{U}_b \ni b$ similarly.

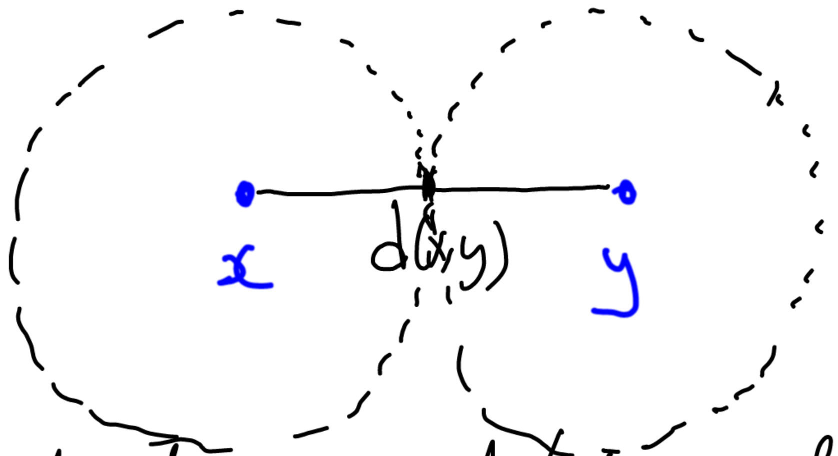
(2) $\mathcal{U}_a$ is open in Y because V_a is open in X and f is continuous.

(3) $\mathcal{U}_a \cap \mathcal{U}_b = f^{-1}(V_a) \cap f^{-1}(V_b) = f^{-1}(V_a \cap V_b) = f^{-1}(\emptyset) = \emptyset$. $\blacksquare$

Prop: Metric topology is Hausdorff.

Pf Let (X,d) be a metric space. Let $x,y \in X$, $x \neq y$. $\implies d(x,y) > 0$

$$r = \frac{d(x,y)}{2}$$



Put $U = B_r(x)$, $V = B_r(y)$

Use the standard argument (triangle inequality) to conclude: $B_r(x) \cap B_r(y) = \emptyset$. $\blacksquare$

Prop: A subspace of a Hausdorff space is Hausdorff.

Pf Let X be a Hausdorff top. space.
Let A be a subset of X, considered with subspace topology: "open in A" means a subset of A of the form $\mathcal{U} \cap A$ where $\mathcal{U}$ is open in X.

Let $a \neq b$, $a, b \in A$.
Then $a, b \in X$ so $\exists\, \mathcal{U}, V$ open in X, $\mathcal{U} \cap V = \emptyset$, $a \in \mathcal{U}$, $b \in V$.

Put $\mathcal{U}' = \mathcal{U} \cap A$, $V' = V \cap A$. Then:

① $\mathcal{U}' \ni a$, $V' \ni b$

② $\mathcal{U}', V'$ are open in A by def. of subspace topology

③ $\mathcal{U}' \cap V' = (\mathcal{U} \cap A) \cap (V \cap A) = \mathcal{U} \cap V \cap A$

$= \emptyset \cap A = \emptyset$. $\blacksquare$

Prop In a Hausdorff space, a point is closed.

"singleton"

Pf Let X be Hausdorff, $x \in X$.

We prove: $\{x\}$ is a closed subset of X.

That is, $X \setminus \{x\}$ is open – proof:

for each $y \in X \setminus \{x\}$, let U_y, V_y be disjoint open neighbourhoods of x and y:

Since $V_y \cap U_y = \emptyset$, $V_y \not\ni x \Rightarrow V_y \subseteq X \setminus \{x\}$

Consider $\bigcup_{y \in X \setminus \{x\}} V_y \supseteq X \setminus \{x\}$ Because each $y \in X \setminus \{x\}$ is covered by V_y

$X \setminus \{x\} \supseteq \bigcup_{y \in X \setminus \{x\}} V_y$ as $V_y \subseteq X \setminus \{x\}$ for $\forall y$

$\boxed{\text{open}}$ as a union of open sets

In conclusion, $X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} V_y$ open. $\blacksquare$