MATH31010 TOPOLOGY AND ANALYSIS

Week 03 lecture A	
SEAtS PIN 079204	

Homeomorphisms (cont'd)

Reminder - Def. of homeomorphism

Let X, Y be top. spaces. A function $f: X \to Y$ is a homeomorphism if (1) f is bijective; (2) f is continuous; (3) $f^{-1}: Y \to X$ is continuous.

* The spaces X and Y are homeomorphic if there exists a homeomorphism $f: X \to Y$. This is an equivalence relation:

$X \xrightarrow[id_X]{\sim} X$ as $id_X = id_X^{-1}$ is continuous; $X \xrightarrow[f]{\sim} Y \Rightarrow Y \xrightarrow[f^{-1}]{\sim} X$;

$X \xrightarrow[f]{\sim} Y \xrightarrow[g]{\sim} Z \Rightarrow X \xrightarrow[g \circ f]{\sim} Z$ as $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

* One of the main problems of topology is to determine which spaces are homeo and which are not

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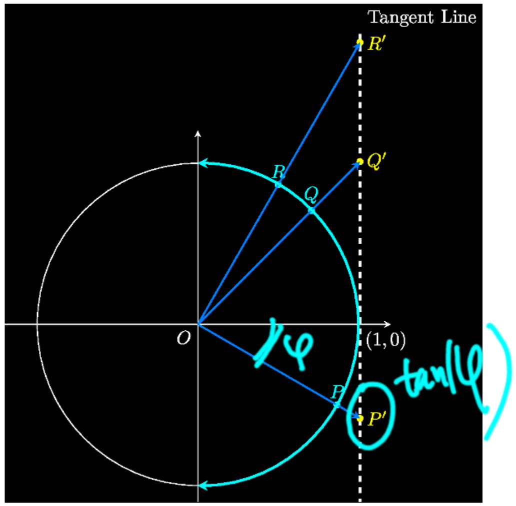
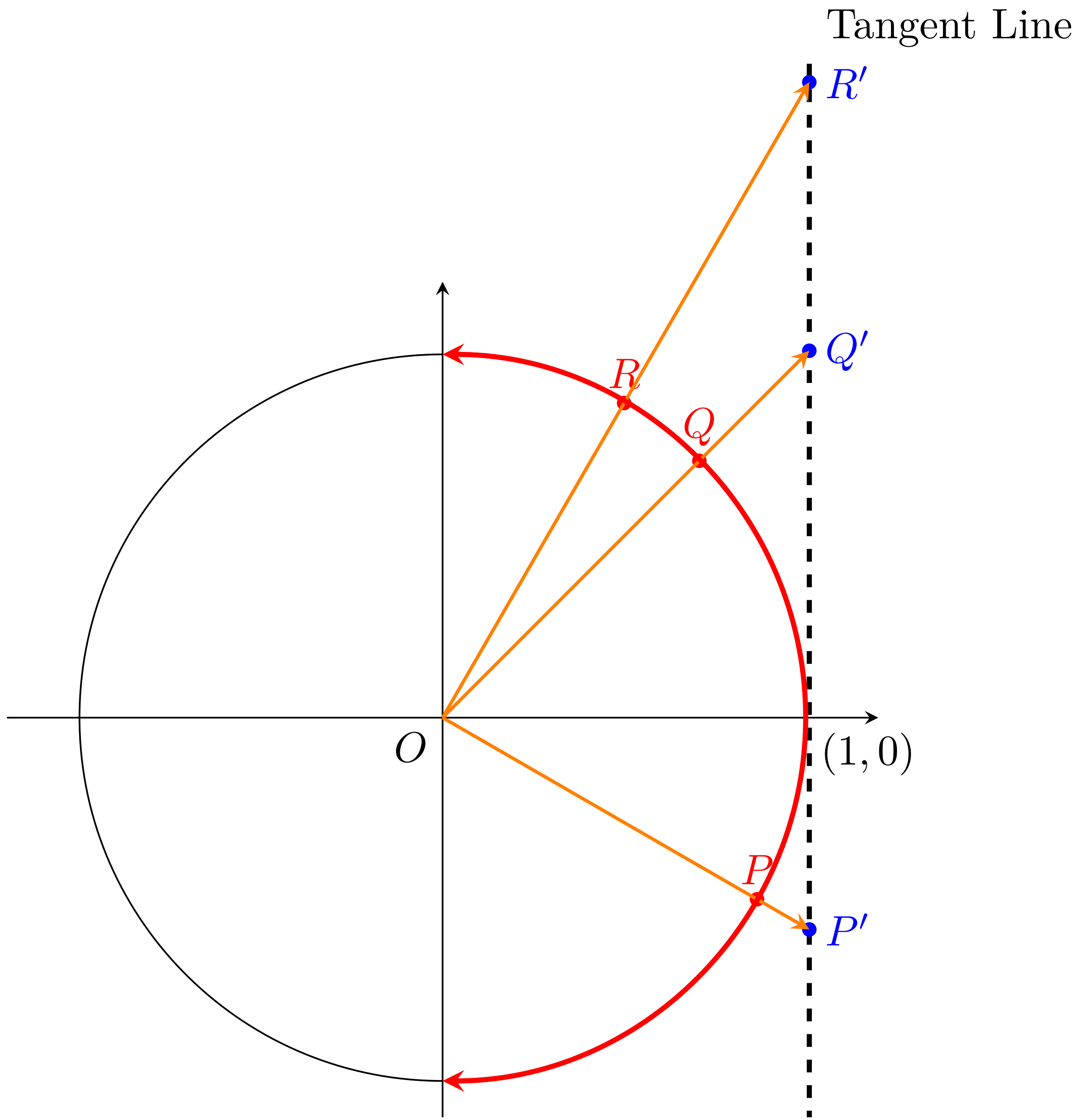
Example. The following spaces (with Euclidian topology) are pairwise homeomorphic:

- the right open half-circle of the unit circle
- the real line
- the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, or $(0,1)$, or (a,b)
- the open half-line $(0,+\infty)$ [Exercise]



$\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a homeomorphism

__But__ are they homeomorphic to



Tangent Line
R'
Q'
R
Q
O
$(1, 0)$
P
P'

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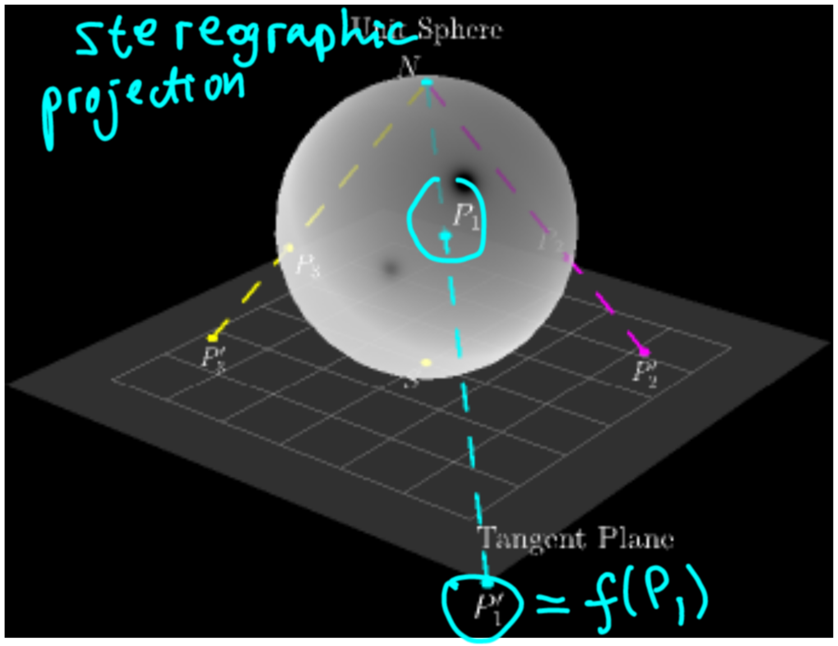
Example: the following are pairwise homeomorphic:

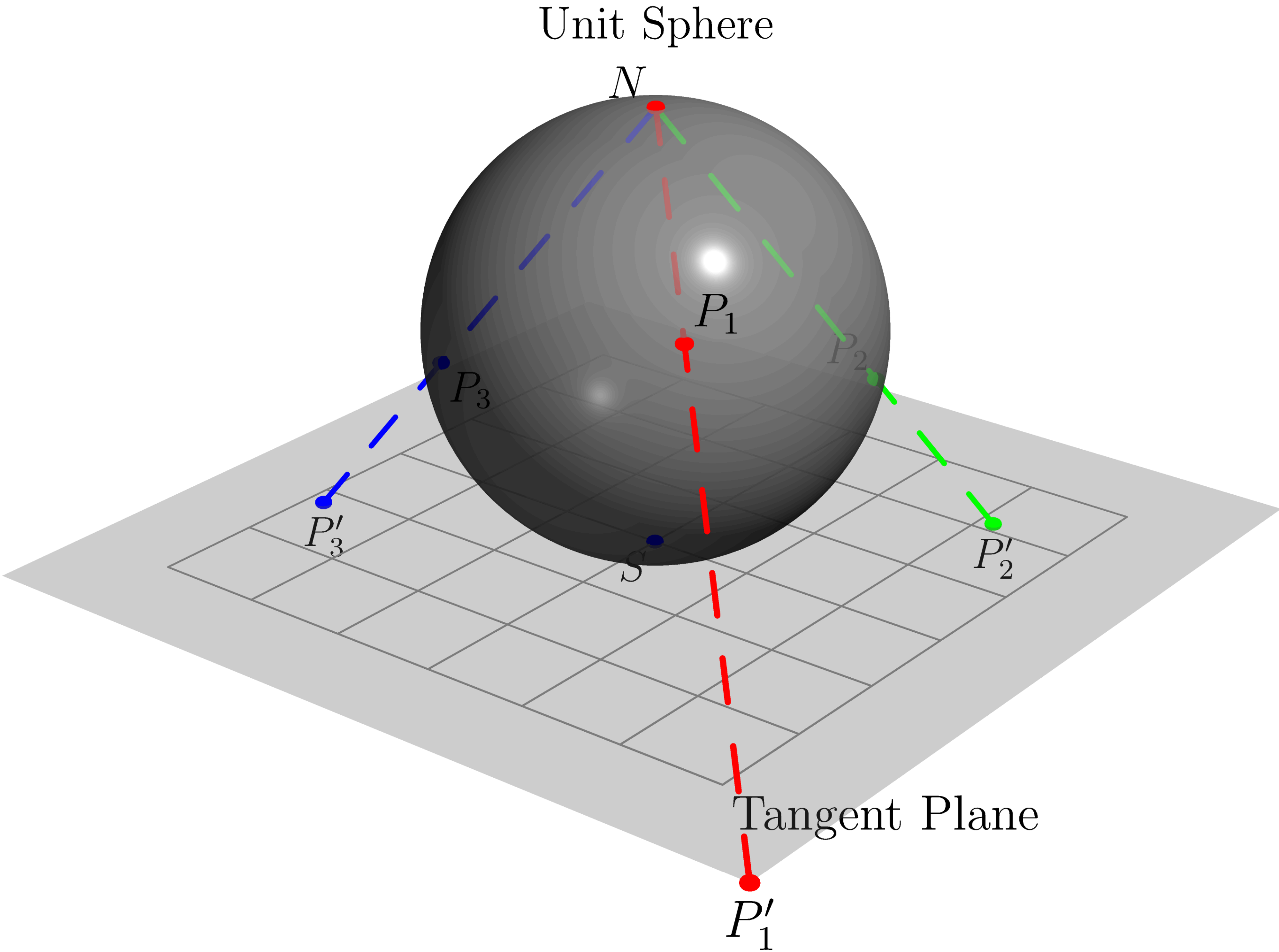
- $\mathbb{R}^2$
- a punctured sphere (sphere $\setminus$ {one point})
- an open half-plane

- an angle without boundary

- an open disc in $\mathbb{R}^2$



Are these homeomorphic to $\mathbb{R}$?



Unit Sphere
N
P_1
P_2
P_3
S
P'_3
P'_2
Tangent Plane
P'_1

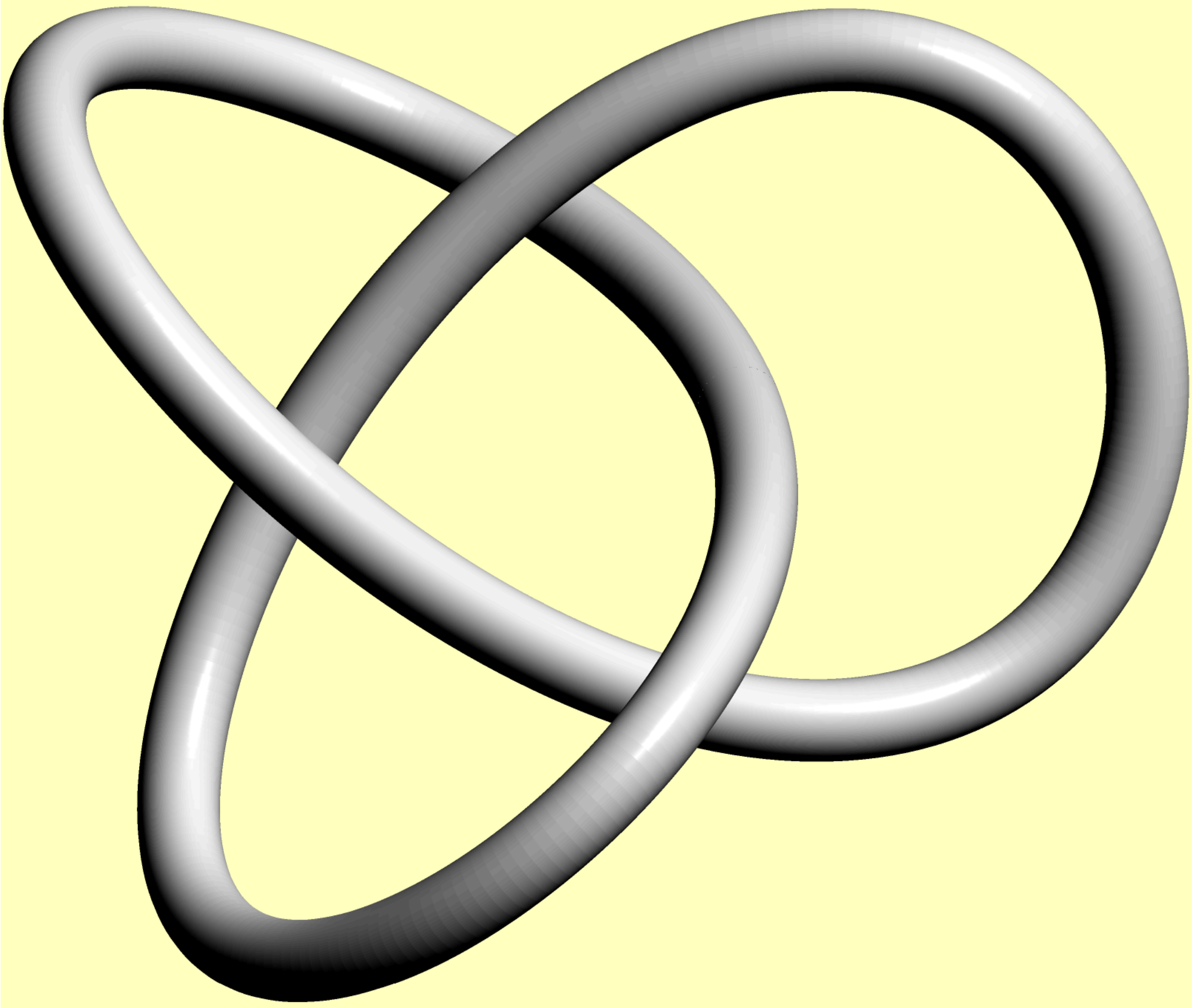
To show that top. spaces X and Y are not homeomorphic, we look for a property of X which must be shared by every space homeomorphic to X, [topological property] but Y does not have this property.

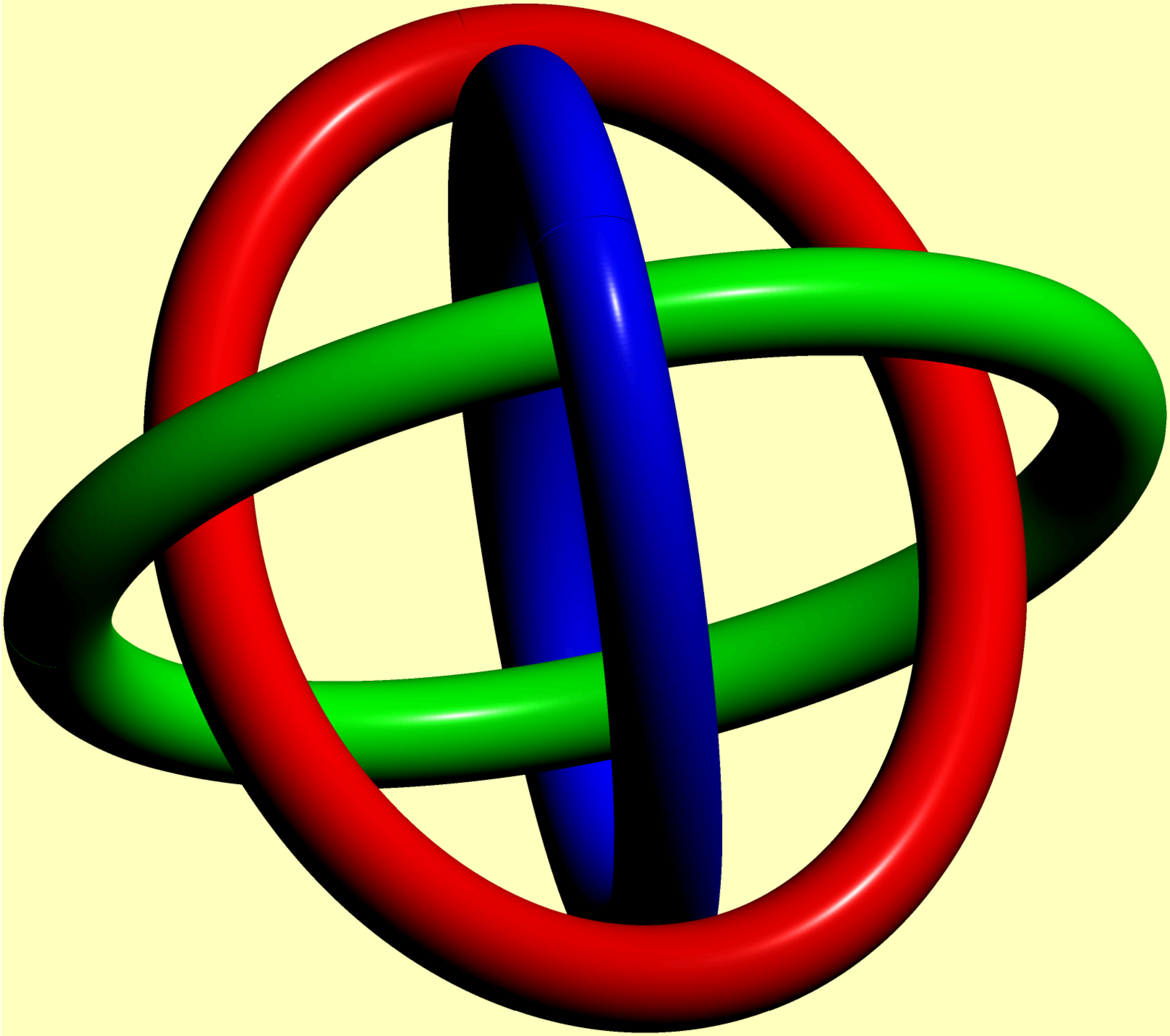
E.g. $X \neq \emptyset$ ← but this is too easy....

– Another main problem of topology:

Given spaces X, Y, can Y be a continuous image of X?

I.e. is there a continuous $f: X \to Y$ s.t. $f(X) = Y$?





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Hausdorff spaces

Def A topological space X is **Hausdorff** if

$$\forall x, y \in X, \quad x \neq y \Rightarrow \exists \text{ open } U, V$$

such that

$$U \ni x, \; V \ni y, \; U \cap V = \emptyset$$

"Two distinct points of the space must have disjoint $\underbrace{\text{open neighbourhoods}}$"

An open nbhd of a point x is any open set which contains x.