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Topology Feedback Quiz, week 2: bases, continuous functions

Open books. 10–15 minutes. Not for credit. To be marked in class.

SEATS 838292

We work with functions $f\colon \mathbb{R} \to \mathbb{R}$.

We will consider three topological spaces:

- $\mathbb{R}_{\text{antidiscrete}}$, the real line with antidiscrete topology , $T = \{\emptyset, \mathbb{R}\}$
- $\mathbb{R}$, the real line with Euclidean topology $\bigcup\{$collection of open intervals$\}$
- $\mathbb{R}_{\text{discrete}}$, the real line with discrete topology All sets are open

Question 1 ♣ Is the collection

$$\{(a,b) : a, b \in \mathbb{R}\}$$

of all intervals a base, or at least an open cover, for each of the three spaces?

- ✗ ◯ base for $\mathbb{R}_{\text{antidiscrete}}$
- ✗ ◯ open cover for $\mathbb{R}_{\text{antidiscrete}}$ (a,b) is NOT OPEN
- ✓ ◯ base for $\mathbb{R}$ by definition
- ✓ ◯ open cover for $\mathbb{R}$ (every base is an open cover!)
- ✗ ◯ base for $\mathbb{R}_{\text{discrete}}$ $\{1\}$ open but not a union of intervals
- ✓ ◯ open cover for $\mathbb{R}_{\text{discrete}}$

Question 2 ♣ Is the collection

$$\{\{p\} : p \in \mathbb{R}\}$$

of all **singletons** a base, or at least an open cover, for each of the three spaces?

- ✗ ◯ base for $\mathbb{R}_{\text{antidiscrete}}$
- ✗ ◯ open cover for $\mathbb{R}_{\text{antidiscrete}}$ $\{1\}$ not open!
- ✗ ◯ base for $\mathbb{R}$
- ✗ ◯ open cover for $\mathbb{R}$ $\{1\}$ not open!
- ✓ ◯ base for $\mathbb{R}_{\text{discrete}}$
- ✓ ◯ open cover for $\mathbb{R}_{\text{discrete}}$

$f: X \to Y$ continuous $\overset{}{\underset{\text{def}}{\Leftrightarrow}}$ $\forall V \underset{\text{open}}{\in} Y$, $f^{-1}(V)$ is open in X

Question 3 ♣ Which functions on $\mathbb{R}^2$ are continuous? Here $\mathbb{R}^2$ has Euclidean topology, and (x,y) denotes a point in $\mathbb{R}^2$.

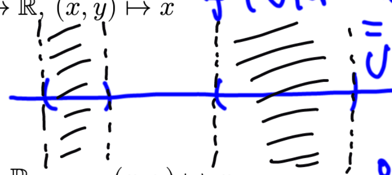
- ✓ ◯ $\mathbb{R}^2 \to \mathbb{R}_{\text{antidiscrete}}$, $(x,y) \mapsto x$

  $f^{-1}(\emptyset) = \emptyset$ open

  $f^{-1}(\mathbb{R}_{\text{antidiscrete}}) = \mathbb{R}^2$ open

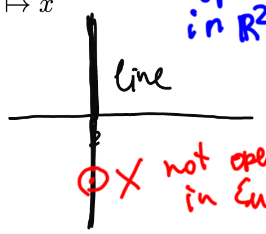
- ✓ ◯ $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$

  $f^{-1}(\bigcup\{\text{open intervals}\}) = \bigcup$ open strips is open in $\mathbb{R}^2$



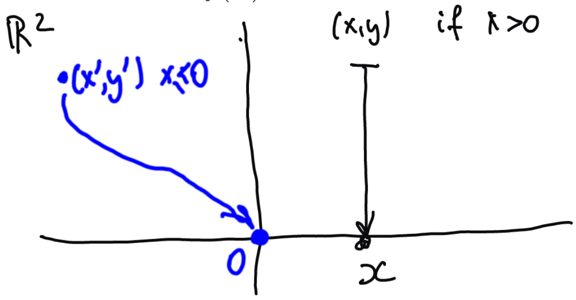
- ✗ ◯ $\mathbb{R}^2 \to \mathbb{R}_{\text{discrete}}$, $(x,y) \mapsto x$

  $f^{-1}(\{2\}) =$ line, open, not open in Euclidean



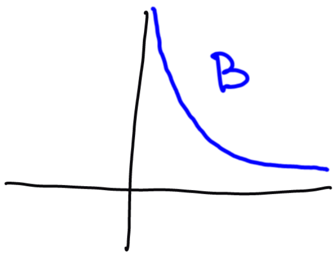
Question 4 Write down an example of a **continuous** function $f\colon \mathbb{R}^2 \to \mathbb{R}$ (Euclidean topology on both) and sets $A, B \subset \mathbb{R}^2$ such that:

- A is open but $f(A)$ is not open;
- B is closed but $f(B)$ is not closed.



$$f((x,y)) = \frac{x + |x|}{2}$$

$A = \mathbb{R}^2$ open, $f(A) = [0, +\infty)$ not open



$B = \{(x, \frac{1}{x}) : x > 0\}$

B is closed in $\mathbb{R}^2$

$f(B) = (0, +\infty)$ not closed in $\mathbb{R}$

Week 2

Exercises (answers at end)

Version 2024/10/04 . To accessible online version of these exercises

Exercise 2.1. (a) Prove that the collection $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, +\infty) : x \in \mathbb{R}\}$ is a topology on the set $\mathbb{R}$ of real numbers.

(b) Prove that the collection $\mathcal{N} = \{\emptyset, \mathbb{R}\} \cup \{[x, +\infty) : x \in \mathbb{R}\}$ is not a topology on the set $\mathbb{R}$. Which axiom(s) of topology is/are not satisfied?

Exercise 2.2. Consider the set $X = \{1, 2\}$ with two points. Describe all possible topologies $\mathcal{T}$ on X. Among the topologies that you describe, identify the discrete topology, the antidiscrete topology and the cofinite topology.

$X = \{1, 2\}$ Subsets:

Must be in $\mathcal{T}$: $\underline{\emptyset}$ $\{1\}$ $\{2\}$ $\underline{X}$

antidiscrete: $\{\emptyset, X\}$ $\{\emptyset, \{2\}, X\}$

$\{\emptyset, \{1\}, X\}$ $\{\emptyset, \{1\}, \{2\}, X\}$ discrete || cofinite

Exercise 2.3. Call a subset A of $\mathbb{R}$ "cocountable" if $A = \emptyset$ or $\mathbb{R} \setminus A$ is finite or countably infinite.

(a) Show that the collection of all cocountable subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$.

(b) Is this topology the same as discrete topology? Antidiscrete topology? Cofinite topology?