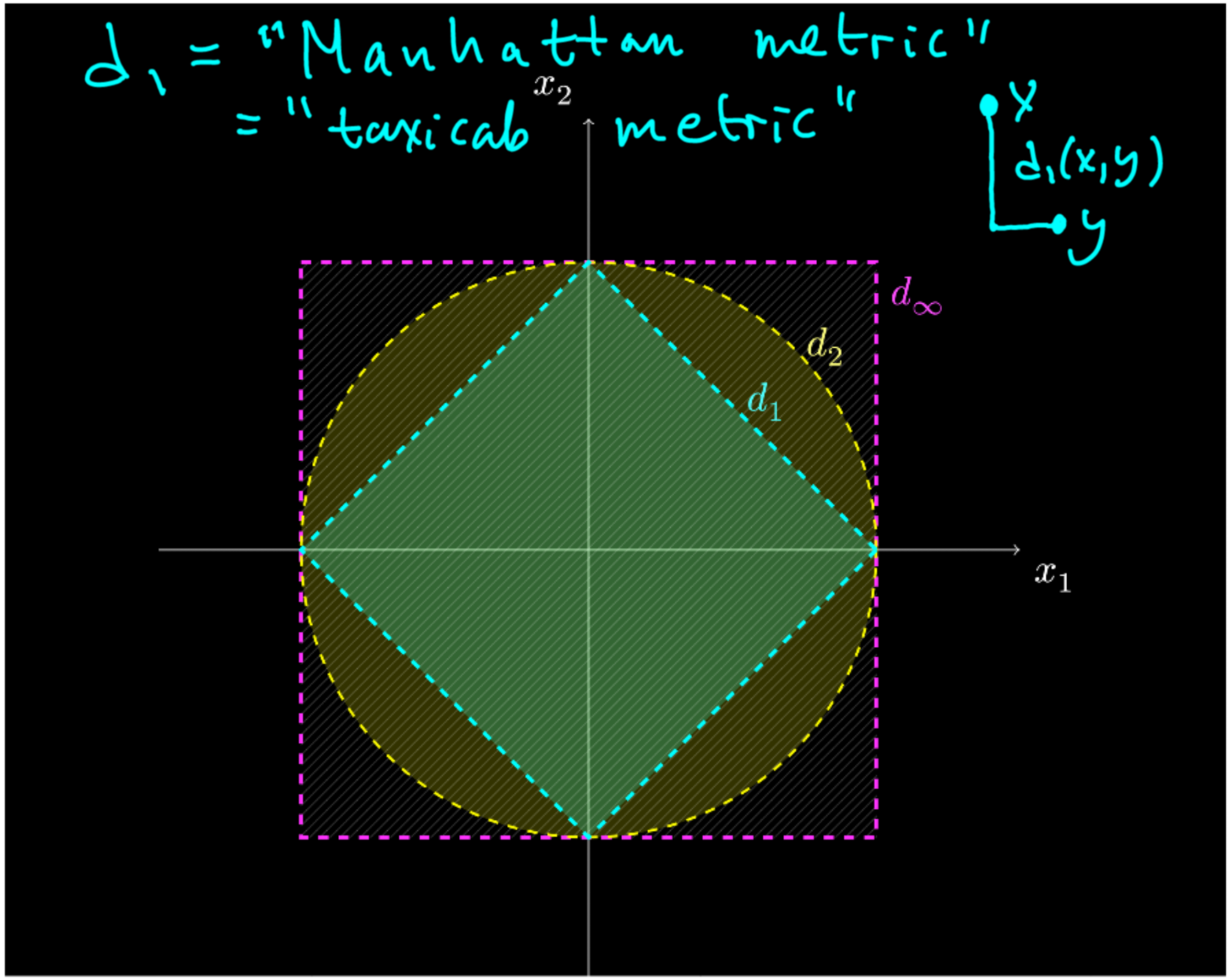
MATH31010 TOPOLOGY AND ANALYSIS

Week 02 **lecture** B

SEAtS PIN 511684



Each one of the metrics d_1, d_2, d_∞ defines its own open balls around each point $\Rightarrow$ different bases for the same Euclidean topology

SEAtS PIN 511684

Continuous functions

DEF Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous** if for every subset V of Y, open in Y, the preimage $f^{-1}(V)$ is open in X.

Rem If X, Y are metric spaces, and the topologies on X, Y are defined by the metric, then $f : X \to Y$ is continuous (topology) $\iff$ $f : X \to Y$ is metric-continuous (the ε-δ definition of continuity)

proved in MATH 21111

SEAtS PIN 511684

REMEMBER:

$$f \text{ continuous} \iff f^{-1}(\text{open}) = \text{open}!$$

DEF Let X be a topological space. A subset F of X is **closed (in X)** if $X \setminus F$ is open in X.

Rem Subsets of X can be:

open but not closed	closed but not open	open and closed	neither open nor closed
$\mathbb{R}$: $(0,1)$	$[0,1]$	$\emptyset$, X	$[\ \ )$

SEAtS PIN 511684

PROP

(a) $\emptyset$, X are closed in X

(b) Arbitrary intersections of closed sets are closed

(c) Finite unions of closed sets are closed.

(sketch)
Proof Take the prop. about open sets and apply the De Morgan Laws to their complements. $\blacksquare$

PROP $f : X \to Y$ continuous $\iff$ the preimage of every closed subset of Y is closed in X.

SEAtS PIN 511684

Proof f continuous $\overset{\text{def}}{\Longleftrightarrow}$

$\forall V \underset{\text{open}}{\subseteq} Y$, $f^{-1}(V)$ is open in X

$\Longleftrightarrow$

$[f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$, see Tutorial$]$ $[$put $F = Y \setminus V]$

$\forall F \underset{\text{closed}}{\subseteq} Y$, $f^{-1}(F)$ is closed in X. $\blacksquare$

PROP $X \xrightarrow{f} Y \xrightarrow{g} Z$ continuous

$\Rightarrow X \xrightarrow{g \circ f} Z$ is continuous.

SEAtS PIN 511684

<u>Proof</u> Need: $\forall W \underset{\text{open}}{\subseteq} Z$,
$(g \circ f)^{-1}(W)$ is open in X.

Easy ~~to~~ see: $\boxed{(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))}$

So if W is open in Z, then $g^{-1}(W)$ open in Y, because g is continuous;
$f^{-1}(\ g^{-1}(W)\ )$ is open in X because f is continuous. $\blacksquare$

SEAtS PIN 511684

EX Let X be an (arbitrary) topological space. The identity function

$$\mathrm{id}_X : X \to X$$
$$x \mapsto x \quad \forall x \in X$$

is continuous. [Proof: $\mathrm{id}_X^{-1}(\underset{\text{open}}{V}) = \underset{\text{open}}{V}$]

EX Let X, Y top. spaces, $y_0 \in Y$ point in Y

$$\mathrm{const}_{y_0} : X \to Y$$
$$x \mapsto y_0$$

is continuous:

$$\mathrm{const}_{y_0}^{-1}(V) = \left.\begin{cases} X, & \text{if } y_0 \in V \\ \emptyset & \text{if } y_0 \notin V \end{cases}\right\} \text{open!}$$

SEAtS PIN 511684

DEF (subspace topology)

Let X be a top. space, let A be a subset of X. Define the following topology on A:

$$V \subseteq A \text{ is } \underline{\text{open in } A} \overset{\text{def}}{\iff} \exists\, U \underset{\text{open}}{\subseteq} X : \quad V = U \cap A$$

This is called "subspace topology" on A. A becomes a subspace of X.

__Exercise__ Show that a "subspace topology" is a topology.

Def Let A be a subspace of X. The function

$$\mathrm{in}_A : A \to X$$
$$a \mapsto a$$

is the inclusion function of A (in X).

PROP in_A is continuous.

Pf Let $V \underset{\text{open}}{\subseteq} X$.

$\mathrm{in}_A^{-1}(V) = V \cap A$ is open in A by Def'n of subspace topology on A.

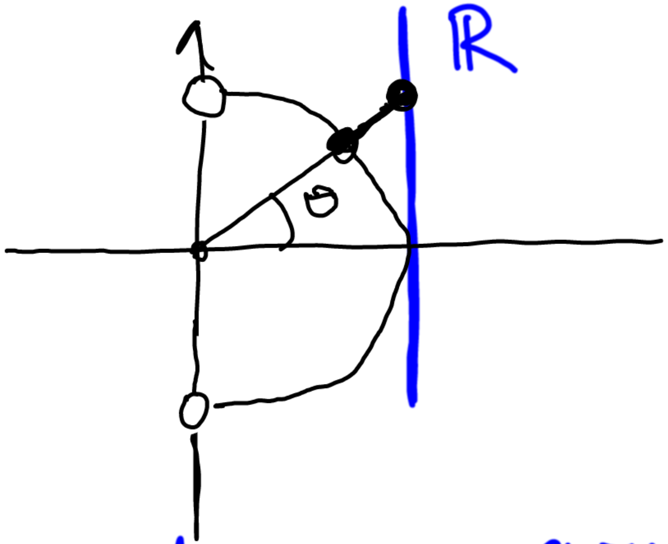
Def Let X, Y be top. spaces. A **homeomorphism** between X and Y is a function $f: X \longrightarrow Y$ such that

* f is bijective
* f is continuous
* f^{-1} (the inverse of f) is continuous.
  $Y \to X$

open interval

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\theta \in$



continuous

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\sim} \mathbb{R}$

HOMEOMORPHISM:

$$\left(-\pi/2, \pi/2\right) \longrightarrow \mathbb{R}$$

$$\theta \longmapsto \tan(\theta)$$

$$\arctan(x) \longleftarrow\!\!\mid\; x$$