MATH31010 TOPOLOGY AND ANALYSIS **Week** 02 **lecture** A

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Metric topology

Let (X, d) be a metric space:

here d is a metric on the set X,

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$$

Def's

Open ball in (X, d): $B_r(x) = \{ y \in X : d(y, x) < r \}$

radius ≥ 0

centre, $x \in X$

An open set in (X, d): a union of some collection of open balls.

THM If (X, d) is a metric space, the collection $\mathcal{T}_d$ of all d-open subsets of X is a topology on X.

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DEF's d is a metric on X

$\Rightarrow \mathcal{T}_d$ is a metric topology (on X)

A top. space $(X, \mathcal{T})$ is metrisable if there exists a metric d on X for which $\mathcal{T}_d = \mathcal{T}$.

$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R} \ \forall i\}$ The Euclidean distance between $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ is

$$d_2(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$$

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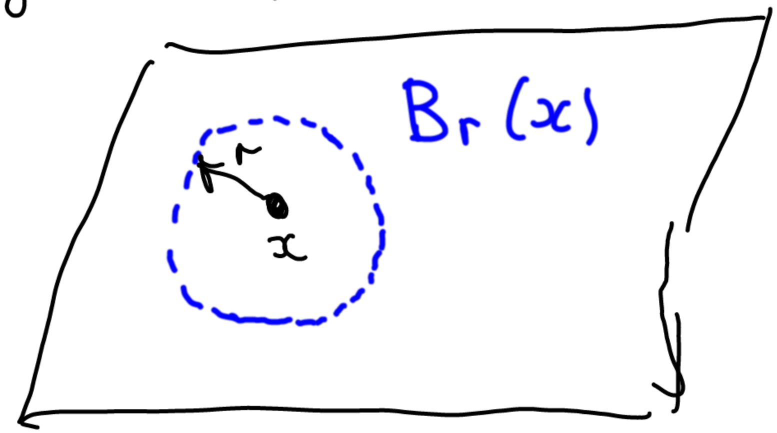
The d_2 makes $\mathbb{R}^n$ a metric space; therefore — see THM — a topological space. Euclidean (topological) space $\mathbb{R}^n$.

Remark 1: any subset $X \subseteq \mathbb{R}^n$ becomes a metric space, hence a topological space.

<u>Ex</u> $\mathbb{R}^2$ Euclidean plane

$B_r(x)$ = open disc around x of radius r



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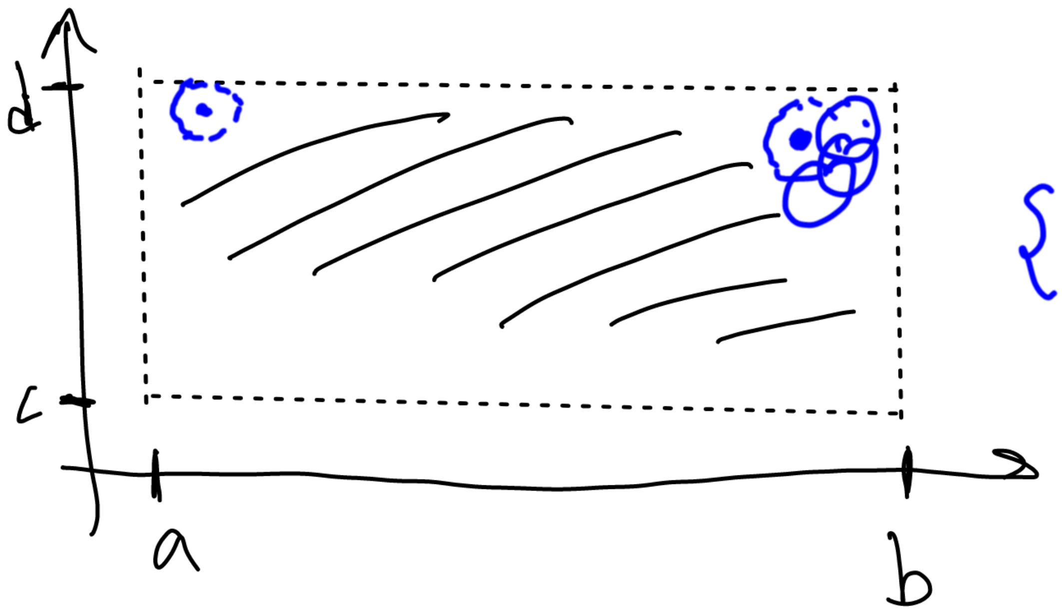

open rectangle

$\{(x_1, x_2) : a < x_1 < b, \; c < x_2 < d\}$

is a union of some collection of open discs

Def $(X, \mathcal{T})$ is an (arbitrary) top. space.

An open cover of X is a collection $\mathcal{C}$ of open subsets of X such that $\bigcup \mathcal{C} = X$.

A base of topology $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X s.t. $\mathcal{T} = \{$all possible unions of subcollections of $\mathcal{B}\}$

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Rem 1 Every base of topology on X is an open cover of X: unions of single-set subcollections of $\mathcal{B}$ are just sets from $\mathcal{B}$ and they must be members of $\mathcal{T}$, hence open; also, X is open (axiom (i)) so $X = \bigcup \mathcal{B}'$, $\mathcal{B}' \subseteq \mathcal{B}$, so $\bigcup \mathcal{B} = X$.

Rem 2 Any $\mathcal{T}$ has a base, e.g. $\mathcal{B} = \mathcal{T}$ (not interesting). However, a metric topology $\mathcal{T}_d$ has base $\mathcal{B} = \{$all d-open balls in $X\}$;

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Rem 3 Are there any interesting bases of the Euclidean topology on $\mathbb{R}^n$, other than Euclidean open balls?

Yes **Reminder** Two metrics d, e on the set X are Lipschitz equivalent if $\exists\ h, k > 0$:

$$\forall x, y \in X \quad h e(x,y) \leq d(x,y) \leq k e(x,y)$$

THM (Math 21111): d, e on X are Lipschitz equivalent metrics $\Longrightarrow \tau_d = \tau_e$.

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Rem on $\mathbb{R}^n$:

metrics

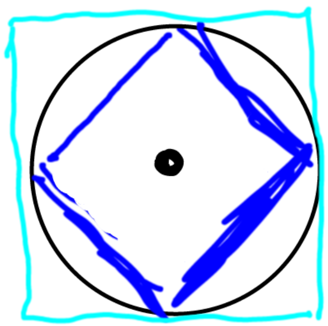
these are Lip. equivalent to d_2

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

$$d_\infty(x,y) = \max_{i=1}^{n} |x_i - y_i|$$

e.g. $\mathbb{R}^2$

unit balls $B_1((0,0))$



d_1 - unit ball

d_2 - unit ball

d_∞ - unit ball

$\boxed{\text{THREE BASES, SAME TOPOLOGY}}$