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Topology Feedback Quiz, week 1: sets and functions
 Open books. 10-15 minutes. Not for credit. To be marked in class.

We work with functions $f: \mathbb{R} \rightarrow \mathbb{R}$.

Recall that the image of a set $A \subseteq \mathbb{R}$ under f is $f(A) = \{f(a) : a \in A\}$, and the preimage of a set $B \subseteq \mathbb{R}$ is $f^{-1}(B) = \{x \in \mathbb{R} : f(x) \in B\}$.

"Is it always true that..." below asks whether the statement is true for all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and all subsets $A, B \subseteq \mathbb{R}$.

Question 1 Is it always true that $f(A \cup B) = f(A) \cup f(B)$?

Yes No, give a counterexample
 $f(A \cup B) = \{y \in \mathbb{R} : (\exists x \in A \cup B : f(x) = y)\}$
 $= \{y : \exists x \in A, f(x) = y\} \cup \{y : \exists x \in B, f(x) = y\}$

Question 2 Is it always true that $f(A \cap B) = f(A) \cap f(B)$?

A: $f(A)$
B: $f(B)$

" $x \in A \cup B$ " \Leftrightarrow
 $\Leftrightarrow (x \in A) \text{ OR } (x \in B)$

$\exists x \in A \cup B : P(x) \Leftrightarrow$
 $\Leftrightarrow (\exists x \in A : P(x)) \text{ OR } (\exists x \in B : P(x))$

$\exists x \in A : P(x)$	$\exists x \in B : P(x)$	LHS	RHS
F	F	F	F
T	F	T	T
F	T	T	T
T	T	T	T

Yes No, give a counterexample

Question 3 Is it always true that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?
 $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
 Both sides are $\{x \in \mathbb{R} : f(x) \in A \cup B\} =$
 $f(x) = x^2$
 $f(\sqrt{-13}) = f(13) = f(f(13))$
 $A = \{-13\}, B = \{13\}, A \cap B = \emptyset$

Question 4 Is it always true that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$?

Yes No, give a counterexample
 $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
 $= \{x \in \mathbb{R} : f(x) \in A\} \cap \{x \in \mathbb{R} : f(x) \in B\}$

Is $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$?

$\{x \in \mathbb{R} : f(x) \in A \cap B\}$
 $= \{x \in \mathbb{R} : f(x) \in A\} \cap \{x \in \mathbb{R} : f(x) \in B\}$

Question 5 Is it always true that $f(\mathbb{R} \setminus A) = \mathbb{R} \setminus f(A)$? $f(x) = 1 \quad \forall x \in \mathbb{R}$
 Yes No, give a counterexample $f(\mathbb{R} \setminus \mathbb{R}) = \emptyset, \mathbb{R} \setminus f(\mathbb{R}) = \mathbb{R} \setminus \{1\}$

Question 6 Is it always true that $f^{-1}(\mathbb{R} \setminus A) = \mathbb{R} \setminus f^{-1}(A)$?
 Yes No, give a counterexample $f^{-1}(\mathbb{R} \setminus A) = \{x \in \mathbb{R} : f(x) \notin A\} = \mathbb{R} \setminus \{x \in \mathbb{R} : f(x) \in A\}$

Question 7 Is it always true that if A is finite, then $f(A)$ is finite?
 Yes No, give a counterexample _____

Question 8 Is it always true that if A is finite, then $f^{-1}(A)$ is finite?
 Yes No, give a counterexample $f(x) = 1 \quad \forall x$
 $f^{-1}(\{1\}) = \mathbb{R}$

Question 9 Is it always true that if A is infinite, then $f(A)$ is infinite?
 Yes No, give a counterexample $f(x) = 1 \quad f(\mathbb{R}) = \{1\}$

Question 10 Is it always true that if A is infinite, then $f^{-1}(A)$ is infinite?
 Yes No, give a counterexample $f(x) = 1$
 $f^{-1}(\underbrace{(2,3)}_{\text{infinite}}) = \emptyset$ finite