IN LECTURE A WE GAVE THE FOLLOWING

DEF (Topological space)

Let X be a set. Suppose $\mathcal{T}$ is a collection of subsets of X such that:

(i) $X \in \mathcal{T}$

(ii) for any subcollection $\mathcal{T}_1$ of $\mathcal{T}$, $\bigcup \mathcal{T}_1$ is in $\mathcal{T}$

(iii) for any $A, B \in \mathcal{T}$, $A \cap B \in \mathcal{T}$

Then $\mathcal{T}$ is called a topology on (the set) X, and $(X, \mathcal{T})$ is a topological space.

* A subset of X which is a member of $\mathcal{T}$ is called open (set). Elements of X are called points.

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1. Is $\emptyset$ open?
2. What about intersections of more than 2 open sets?

PROP Let $(X, \mathcal{T})$ be a topological space. Then

(a) X and $\emptyset$ are open;

(b) arbitrary unions of open sets are open;

(c) finite intersections of open sets are open.

Sketch of proof. (a) X is open by axiom (i).
$\emptyset$ = the union of an empty collection of open sets, so open by axiom (ii). (b) = axiom (ii).

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(c): Induction. Base case $n=2$: axiom (iii)

Inductive step: $n-1 \longrightarrow n \quad (n \geq 3)$

$$U_1 \cap U_2 \cap \cdots \cap U_n = (\underbrace{U_1 \cap U_2}_{\text{(open)}}) \cap U_3 \cap \cdots \cap U_n$$

intersection of $n-1$ open sets, open by Inductive hypothesis. ■

Example ($X = \emptyset$, $\tau = \{\emptyset\}$) is a topological space.

Remark ~~An infinite~~ The intersection of infinitely many open sets may not be open.
See below for counterexamples.

Remark More than 1 topology may exist 175026 on the same set X.

DEF Let X be a set.

1. The **discrete topology** on X: **all** subsets of X are open.
2. The **antidiscrete topology** on X: the only open sets are $\emptyset$ and X.
3. The **cofinite topology**: open sets are $\emptyset$ and all U where $X \setminus U$ is a finite set.

DEF If $A \subseteq X$, the subset $X \setminus A$ of X is called the **complement** of A (in X).

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Lemma Let $A, B \subseteq X$. If $A \subseteq B \iff (X \setminus B) \subseteq (X \setminus A)$.

Lemma (De Morgan laws)

Let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of X.

$$X \setminus \left(\bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$$

$$X \setminus \left(\bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$$

PROP Let τ be the collection of subsets of X which consists of $\emptyset$ and all subsets with finite complement in X. Then τ is a topology on X.

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Pf (i) $X \in \mathcal{T}$ because X has complement, $\emptyset$.

(ii) Let $\mathcal{F}$ be a subcollection of $\mathcal{T}$. Need to prove: $\bigcup \mathcal{F} \in \mathcal{T}$.

<u>CASE 1</u> All sets in $\mathcal{F}$ are empty.
Then $\bigcup \mathcal{F} = \emptyset \in \mathcal{T}$.

<u>CASE 2</u> There is a non-empty set $\mathcal{U} \in \mathcal{F}$.

$$\mathcal{U} \subseteq \bigcup \mathcal{F} \Rightarrow$$

$$(X \setminus \bigcup \mathcal{F}) \subseteq X \setminus \mathcal{U} \leftarrow \text{finite set}$$

Lemma

$\Rightarrow X \setminus \bigcup \mathcal{F}$ is finite

$\Rightarrow \bigcup \mathcal{F} \in \mathcal{T}$.

(iii) Let $\mathcal{U}, V \in \mathcal{T}$. <u>CASE 1</u>: $\mathcal{U} = \emptyset$ or $V = \emptyset$

Then $\mathcal{U} \cap V = \emptyset \Rightarrow \mathcal{U} \cap V \in \mathcal{T}$.

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<u>CASE 2</u> $\mathcal{U} \neq \emptyset \Rightarrow X \setminus \mathcal{U}$ is finite

$V \neq \emptyset \Rightarrow X \setminus V$ is finite

$$X \setminus (\mathcal{U} \cap V) \underset{\text{De Morgan}}{=} \underbrace{(X \setminus \mathcal{U})}_{\text{finite}} \cup \underbrace{(X \setminus V)}_{\text{finite}} \text{ is finite}$$

$\Rightarrow \mathcal{U} \cap V \in \mathcal{T}$. ∎

<u>DEF</u> (Metric topology): Let (X, d) be a metric space.

$\rightsquigarrow$ We will ~~define~~ consider the metric topology $\mathcal{T}_d$ on X.