# Week 1

# Topology: basic definitions and examples

Version 2024/11/01 To accessible online version of this chapter

The year-long MATH31010 Topology and Analysis course will consist of **three parts:**

I. Introduction to Topology, lectured by Yuri Bazlov;
II. Introduction to Functional Analysis, lectured by Yotam Smilansky;
III. Further topics in topology and analysis, lectured by Donald Robertson.

You are reading **Part I notes** which are being developed to reflect the content of the course as taught in the 2024/25 academic year. Questions and comments on these lecture notes should be directed to Dr Yuri Bazlov at `Yuri.Bazlov@manchester.ac.uk`.

**Textbooks:** some proofs will follow the book [Sutherland] or [Armstrong], see comments at end of each chapter. Overall organisation of the material differs from either book.

**Use of generative AI in these notes:** by way of an experiment, some diagrams in these notes will use source code generated with the help of artificial intelligence (AI). An acknowledgment will be provided via an [AI] link next to the diagram.

AI is an evolving set of technologies which utilise applications of pure mathematics, including topology (example: topological data analysis). It seems especially fitting that generative AI can now help us visualise definitions and proofs from the Topology and Analysis course.

## An informal overview

Many processes in nature and in industry are modelled by continuous functions. The notion of "continuous" was defined for functions $f\colon X \to Y$ where

- $X$, $Y$ are subsets of $\mathbb{R}$ (B. Bolzano, A.-L. Cauchy, first half of the 19$^{\text{th}}$ century) — as discussed in MATH11121 *Mathematical Foundations and Analysis*;
- $X$, $Y$ are metric spaces (M. Fréchet, early 20$^{\text{th}}$ century) — as discussed in MATH21111 *Metric Spaces*.

Yet some mathematical situations expect continuous functions defined between

- sets with a large class of metrics and no single preferred metric, or
- sets where no metric exists.

An example of a set where no natural metric may exist is an algebraic curve, or more generally an algebraic variety, over a field other than $\mathbb{R}$ or $\mathbb{C}$. Algebraic geometry, a branch of mathematics which was expanding and becoming more formal in early 20$^{\text{th}}$ century, required a rigorous theory of continuous functions between such objects. A nice class of algebraic curves are elliptic curves, which today have extensive applications in many fields, including number theory and cryptography.

Thus, **an important goal of topology is to define "continuous" without a metric.** For this, the sets $X$ and $Y$ need to be equipped with a structure of a **topological space** defined by purely set-theoretic axioms. This approach was developed in the 20$^{\text{th}}$ century work of F. Riesz and F. Hausdorff. (We give the formal definitions after this introduction.) Every metric space is automatically a topological space — you had a glimpse of that in MATH21111 Metric Spaces, where the notion of an *open set* was introduced — but there are topological spaces which cannot be defined by means of a metric.

Two topological spaces $X$, $Y$ are considered equivalent, or **homeomorphic**, if there are bijective functions $f\colon X \to Y$ and $g\colon Y \to X$ which are inverse to each other and are both continuous. This notion is new even for metric spaces: two metric spaces might have very different metrics, but still be homeomorphic. A simple example, see Figure 3.1, shows that the real line $\mathbb{R}$ is homeomorphic to its subset, the open interval $(-\pi/2, \pi/2)$. Yet $\mathbb{R}$ is complete and unbounded metric space, whereas $(-\pi/2, \pi/2)$ is bounded but not complete.

**Topology is concerned with finding properties of a topological space $X$ which are necessarily shared by all spaces homeomorphic to $X$.** Such properties are called **topological properties** for short. The above example tells us that boundedness, and completeness, are **not** topological properties of a metric space.

Especially sought after are topological properties which persist in any continuous image of the space $X$: that is, $f(X)$ where $f$ is continuous, but $f$ is not necessarily a homeomorphism (i.e., $f$ may not be injective, or the inverse of $f$ may not be continuous). In this course, we will study two such important properties: "$X$ is compact" and "$X$ is connected".

The result which states "if $X$ is compact and $f$ is continuous, then $f(X)$ is compact" is a vast generalisation of the theorem from Mathematical Foundations and Analysis which says that a continuous function on a closed bounded interval in $\mathbb{R}$ is bounded and attains its maximum and minimum value. (This was mentioned in MATH21111 Metric Spaces.) We can use this result in many more situations, for example to show that there is no surjective continuous map from a sphere to $\mathbb{R}^2$.

The result which states "if $X$ is connected and $f$ is continuous, then $f(X)$ is connected" is, in turn, a generalisation of the Intermediate Value Theorem from Mathematical Foundations and Analysis. Again, we can apply it to many more situations, for example to give a rigorous proof that the shapes $\bigcirc$ (circle) and $\infty$ (figure of eight) are not homeomorphic.

But the true power of topology lies in its ability to apply the same principles to simple spaces (the circle, $\mathbb{R}^2$ etc) and to vastly more complicated objects such as infinite dimensional normed spaces. Hopefully you will see some of the workings of topology in infinite dimensions in parts II and III of the course.

# Fundamentals of sets

Definitions and axioms in topology are expressed in the language of set theory. We need to be able to speak this language. In this section, we recall fundamental notions from set theory and introduce some notation to be used throughout the course.

> **Notation: set, element, collection.**
>
> **Sets** will be denoted by capital letters $A$, $B$, $C$, ...
>
> **Elements** of a set will typically be written as small letters: $a, b, c \in A$ means that $a$, $b$ and $c$ are elements of the set $A$.
>
> A **collection** (= **family**) of sets is a finite or infinite list of sets. Collections will be denoted by script letters such as $\mathscr{E}$, $\mathscr{F}$, $\mathscr{G}$, ... Sets in a collection may be indexed by some index set: for example, $\mathscr{F} = \{A_\alpha : \alpha \in I\}$ where $I$ is an index set.

It is important to distinguish between elements of a set $A$ and subsets of a set $A$. Recall that $B$ is a **subset** of $A$ if every element of $B$ is also an element of $A$:

$$B \subseteq A \quad \overset{\text{def}}{\iff} \quad \forall x \in B,\ x \in A.$$

We will also consider **subcollections:** a collection $\mathscr{G}$ of sets may be a subcollection of a collection $\mathscr{F}$.

While sets, subsets and elements appear in mathematics of all levels and styles, collections (especially infinite collections) of sets tend to occur in advanced pure mathematical texts. To familiarise ourselves with collections, let us look at simple examples built from subsets of $\mathbb{R}$:

- $\mathscr{F} = \{A\}$, a collection of just one set.
- $\mathscr{G} = \{A, A\}$, a collection of two identical sets. We do not insists that all sets in a collection are different from each other — repetitions are allowed.
- $\mathscr{C} = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$, a countable collection of open intervals in $\mathbb{R}$.
- $\mathscr{U} = \{(0, x) : x \in \mathbb{R},\ x > 0\}$, an uncountable collection of open intervals in $\mathbb{R}$.
- $\mathscr{E} = \emptyset$, an empty collection of subsets of $\mathbb{R}$.

Two common operations can be applied to a collection of subsets of some universal set $X$:

> **Definition: union and intersection of a collection.**
>
> Let $X$ be a set and $\mathscr{F}$ be a collection of subsets of $X$. The **union** of $\mathscr{F}$ is the set
>
> $$\bigcup \mathscr{F} = \{x \in X : \exists A \in \mathscr{F},\ x \in A\}.$$
>
> The **intersection** of $\mathscr{F}$ is the set
>
> $$\bigcap \mathscr{F} = \{x \in X : \forall A \in \mathscr{F},\ x \in A\}.$$

When a collection is finite, or when it is indexed by an index set, alternative notation is often used for the union and intersection of the collection.

> **Notation: variants of notation for unions and intersections.**
>
> | | | |
> |---|---|---|
> | $\mathscr{F} = \{A, B\}$ | $\Rightarrow$ | we can write $\bigcup \mathscr{F} = A \cup B$. |
> | $\mathscr{F} = \{A_1, A_2, \dots, A_n\}$, a collection of $n$ sets | $\Rightarrow$ | $\bigcup \mathscr{F} = A_1 \cup A_2 \cup \dots \cup A_n$. |
> | $\mathscr{F} = \{A_i : i \in \mathbb{N}\}$, a countable collection | $\Rightarrow$ | $\bigcup \mathscr{F} = \bigcup_{i=1}^{\infty} A_i$. |
> | $\mathscr{F} = \{A_\alpha : \alpha \in I\}$ for some index set $I$ | $\Rightarrow$ | $\bigcup \mathscr{F} = \bigcup_{\alpha \in I} A_\alpha$. |
>
> *The above conventions also apply to* $\bigcap$ *in place of* $\bigcup$.

For practice, we work out the unions and intersections of collections of subsets of $\mathbb{R}$ in the example above. We use the notation $\mathbb{R}_{>0}$ for the set of all positive reals.

- $\mathscr{F} = \{A\}$ where $A \subseteq \mathbb{R}$: $\bigcup \mathscr{F} = \bigcap \mathscr{F} = A$.
- $\mathscr{G} = \{A, A\}$: $\bigcup \mathscr{G} = \bigcap \mathscr{G} = A$.
- $\mathscr{C} = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$: $\bigcup_{n=1}^{\infty} (0, \frac{1}{n}) = (0, 1)$, $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$.
- $\mathscr{U} = \{(0, x) : x \in \mathbb{R}_{>0}\}$: $\bigcup_{x>0} (0, x) = \mathbb{R}_{>0}$, $\bigcap_{x>0} (0, x) = \emptyset$.
- $\mathscr{E} = \emptyset$, an empty collection of subsets of $\mathbb{R}$: $\bigcup \mathscr{E} = \emptyset$, $\bigcap \mathscr{E} = \mathbb{R}$.

Note that in the last example, the intersection of an empty collection of subsets of $X$ is $X$, the universal set. This follows logically, since the condition "$\forall A \in \emptyset$, $x \in A$" in the definition of intersection holds for all $x$.

Another way to see that an empty collection has intersection equal to the universal set is to use the De Morgan laws. We will recall the De Morgan laws soon.

## Topology: definition and examples. Open sets

Although the word **"topology"** may mean the area of mathematics studied in this course, we say **"a topology"** to refer to a collection of sets described in the following definition.

**Definition: a topology; topological space; point; open set.**

Let $X$ be a set. Suppose $\mathscr{T}$ is a collection of subsets of $X$ such that

(i) $X \in \mathscr{T}$;

(ii) for every subcollection $\mathscr{T}_1$ of $\mathscr{T}$, the set $\bigcup \mathscr{T}_1$ belongs to $\mathscr{T}$;

(iii) if $A \in \mathscr{T}$ and $B \in \mathscr{T}$, then $A \cap B \in \mathscr{T}$.

Then $\mathscr{T}$ is called **a topology** on the set $X$, and the pair $(X, \mathscr{T})$ is a **topological space.** Elements of $X$ are called **points.** Subsets of $X$ which belong to the topology, i.e., belong to the collection $\mathscr{T}$, are called **open sets.**

We may say "$X$ is a topological space" instead of "$(X, \mathscr{T})$ is a topological space" if it is clear which topology $\mathscr{T}$ is used. (Keep in mind that more than one topology may be defined on the same set $X$.)

We arrive at our first explicit (but trivial!) example of a topological space. It is easy to see that axioms (i)–(iii) hold, as all unions and intersections are equal to the empty set:

**Example: topology on the empty set.**

Let $X = \emptyset$, the empty set. Consider the collection $\mathscr{T} = \{\emptyset\}$ — this collection consists of just one set. The pair $(\emptyset, \{\emptyset\})$ is a topological space.

The following properties of open sets are an easy consequence of the definition of topology. In fact, these properties are equivalent to the axioms of topology:

**Proposition 1.1: properties of open sets.**

If $(X, \mathscr{T})$ is a topological space, then

(a) $X$ and $\emptyset$ are open,

(b) arbitrary unions of open sets are open,

(c) finite intersections of open sets are open.

*Sketch of proof.* (a) $X$ is open by axiom (i); $\emptyset$ is the union of an empty collection of open sets so is open by axiom (ii).

(b) is just axiom (ii) of topology.

For (c), we need to assume that $U_1, U_2, \dots, U_n$ are open sets, and to show that $U_1 \cap \dots \cap U_n$ is open. This is shown by induction where the base case $n = 2$ is axiom (iii) of topology, and the inductive step is done by writing

$$U_1 \cap \dots \cap U_n = (U_1 \cap U_2) \cap U_3 \cap \dots \cap U_n$$

as the intersection of $n - 1$ open sets. □

**Alert: arbitrary intersections of open sets.**

Only intersections of **finitely many** open sets are guaranteed to be open. The intersection of an **infinite** collection of open sets may not be open. Counterexamples will be seen in subsequent lectures and in the tutorial.

In general, there exist many topologies on a given set $X$. We give simple examples of topologies which can be introduced on any set.

**Definition: discrete, antidiscrete and cofinite topologies.**

Let $X$ be a set.

- The **discrete topology** on $X$ is the topology where all subsets of $X$ are open.
- The **antidiscrete topology** on $X$ is where the only open sets are $\emptyset$ and $X$.
- The **cofinite topology** on $X$: $U \subseteq X$ is open iff $U = \emptyset$ or $X \setminus U$ is a finite set.

**Exercise.** Show that the discrete topology is a topology. That is, the collection of **all** subsets of $X$ satisfies axioms (i)–(iii) of topology. Do the same for the antidiscrete topology.

To show that the cofinite topology is a topology, we carefully work with complements.

**Definition: complement.**

Let $X$ be a set and $A$ be a subset of $X$. The subset $X \setminus A$ of $X$ is called the **complement** of the set $A$ in $X$.

For the next result, we need a simple fact about complements (proof: exercise).

**Lemma 1.2: taking the complement reverses inclusion.**

If $A, B$ are subsets of $X$, then $A \subseteq B \iff (X \setminus B) \subseteq (X \setminus A)$. □

We also need the De Morgan laws (proof: exercise). The proof of the next Lemma is an exercise and can be found in the literature.

**Lemma 1.3: the De Morgan laws.**

Let $\{A_\alpha : \alpha \in I\}$ be an arbitrary family of subsets of $X$. Then
$X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$ and $X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$.

(Note how $\bigcup$ changes to $\bigcap$ and vice versa.) In short, the De Morgan laws say

**the complement of a union is the intersection of complements,**
and
**the complement of an intersection is the union of complements.**

We now use lemma 1.2 and the De Morgan laws to prove

**Proposition 1.4: the cofinite topology is a topology.**

The collection $\mathscr{C}$ which consists of the empty set and all subsets of $X$ with finite complement is a topology on the set $X$.

*Proof.* We show that $\mathscr{C}$ satisfies axioms (i)–(iii) from the definition of topology.

(i) $X$ has complement $\emptyset$, and $\emptyset$ is finite, so $X \in \mathscr{C}$.

(ii) Let $\mathscr{F}$ be some collection of sets from $\mathscr{C}$. If all sets in $\mathscr{F}$ are empty, then $\bigcup \mathscr{F} = \emptyset \in \mathscr{C}$.

Otherwise, take a non-empty set $U \in \mathscr{F}$. Then $U$ must have finite complement, and $U \subseteq \bigcup \mathscr{F}$, so by lemma 1.2, $X \setminus \bigcup \mathscr{F} \subseteq X \setminus U$. Yet $X \setminus U$ is a finite set, and all subsets of a finite set are finite. Hence the complement of $\bigcup \mathscr{F}$ is finite, proving that $\bigcup \mathscr{F}$ is in $\mathscr{C}$.

(iii) Suppose $U, V \in \mathscr{C}$. If one of $U, V$ is an empty set, then $U \cap V = \emptyset \in \mathscr{C}$.

Otherwise, $U$ and $V$ are non-empty, and since they are in $\mathscr{C}$, $U$ and $V$ must have finite complements. Then by the De Morgan laws $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$. Thus, $U \cap V$ has a finite complement (a union of two finite sets) and so $U \cap V \in \mathscr{C}$. $\square$

## References for the week 1 notes

Both [Armstrong] and [Sutherland] use the terms **collection** and **family** interchangeably.

Our definition of a **topological space** is the same as [Armstrong, Definition (2.1)]. Note that [Sutherland, Definition 7.1] insists on $X$ being non-empty, but we do not require this.

The **De Morgan laws,** Lemma 1.3: see for example [Willard, Theorem 1.4].

The **antidiscrete topology** is called **indiscrete** in [Armstrong, Problem 29] and [Sutherland, Example 7.5]. The proof that **cofinite topology** is a topology in Proposition 1.4 solves [Sutherland, Exercise 7.5].

# Week 2

# Metric topology. Open covers and bases. Subspace topology. Continuous functions

Version 2024/11/15 To accessible online version of this chapter

Discrete, antidiscrete and cofinite topology on a set $X$, which we have introduced, are good for constructing simple counterexamples, yet they rarely lead to deep constructions in topology or help in applications of topology to other areas of mathematics and physics.

We will now connect abstract topology to the theory of metric spaces, studied in MATH21111. Metric spaces will provide us with a very rich class of examples of topological spaces.

## Metric topologies. Euclidean topologies

Here is an **equivalent** definition of an open set given in MATH21111 *Metric spaces*.

> **Definition: open balls and open sets in a metric space**.
>
> Let $(X, d)$ be a metric space. The **open ball** of radius $r > 0$ around a point $x \in X$ is the set $B_r(x) = \{y \in X : d(y, x) < r\}$.
>
> A $d$-**open set** in $X$ is a union of open balls.

We quote a key result proved in MATH21111:

**Theorem 2.1: metric-open sets in a metric space form a topology.**

If $(X, d)$ is a metric space, the collection $\mathscr{T}_d$ of all $d$-open sets in $X$ is a topology on $X$. □

Topologies arising from metrics deserve a special definition:

**Definition: metric topology, metrisable topological space.**

The topology $\mathscr{T}_d$, where $d$ is a metric on a set $X$, is called a **metric topology.**
A topological space $(X, \mathscr{T})$ is **metrisable,** if there exists a metric $d$ on $X$ such that $\mathscr{T} = \mathscr{T}_d$.

We now introduce what is arguably the most frequently used class of metric spaces and metric topologies. Let $n \geq 1$, and recall that $\mathbb{R}^n$ is the set of $n$-tuples $(x_1, \dots, x_n)$ of real numbers. The **Euclidean metric**

$$d_2((x_1, \dots, x_n), (y_1, \dots, y_n)) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

makes $\mathbb{R}^n$ a metric space.

**Definition: Euclidean topology, Euclidean space.**

The metric topology on $\mathbb{R}^n$ defined by the Euclidean metric is called the **Euclidean topology** and makes $\mathbb{R}^n$ the $n$-dimensional **Euclidean (topological) space.**

Furthermore, arbitrary subsets of $\mathbb{R}^n$ also become topological spaces:

**Example: every subset of a Euclidean space is a topological space.**

If $X \subset \mathbb{R}^n$, the Euclidean metric makes $X$ a metric space hence a topological space.

We will study subspace topology in more detail in the next chapter.

## A base of a topology. Open covers

The definition of an open set in a metric space (a union of open balls) motivates two notions which apply to arbitrary topologies.

**Definition: open cover; base of a topology**.

Let $(X, \mathscr{T})$ be a topological space.

An **open cover** of $X$ is a collection $\mathscr{C}$ of open subsets of $X$ whose union is $X$: $\bigcup \mathscr{C} = X$.

A **base** of the topology $\mathscr{T}$ is a collection $\mathscr{B}$ of subsets of $X$ such that $\mathscr{T}$ consists of unions of all subcollections of $\mathscr{B}$.

**Remark 1:** every topology has at least one base. For example, the whole collection $\mathscr{T}$ is a base for $\mathscr{T}$. But smaller bases are usually more interesting.

**Remark 2:** a base $\mathscr{B}$ of the topology on $(X, \mathscr{T})$ must be an open cover. Indeed, for every set $U \in \mathscr{B}$, the union $\bigcup\{U\}$ of the single-set subcollection $\{U\}$ of $\mathscr{B}$ must belong to $\mathscr{T}$. That is, every set $U \in \mathscr{B}$ is an open set.

Moreover, $X \in \mathscr{T}$ by axiom (i) of topology, and $\mathscr{B}$ is a base, so $X$ must be a union of some subcollection of $\mathscr{B}$. Hence $\bigcup \mathscr{B} \supseteq X$. On the other hand, a union of subsets of $X$ is a subset of $X$, so $\bigcup \mathscr{B} \subseteq X$. Thus, $\bigcup \mathscr{B} = X$.

We have shown that all sets in $\mathscr{B}$ are open, and that the union of all sets in $\mathscr{B}$ is $X$. Therefore, $\mathscr{B}$ is an open cover of $X$.

**Remark 3:** although every base is an open cover, not every open cover is a base. For example, the Euclidean space $\mathbb{R}^n$ has open cover $\{\mathbb{R}^n\}$, but the only unions of subcollections of this cover are $\emptyset$ and $\mathbb{R}^n$. This does not exhaust open subsets of $\mathbb{R}^n$, which has infinitely many open subsets.

**Example 2.2: open balls form a base of a metric topology**.

By definition, a metric topology $\mathscr{T}_d$ has base $\mathscr{B} = \{\text{all open balls in the metric } d\}$.

Does the Euclidean topology on $\mathbb{R}^n$ have other bases? Yes, plenty of other bases are possible.

First of all, we note that a metric $d$ which defines a given metrisable topology on $X$ may not be unique. Recall from MATH21111 that two metrics $d$ and $e$ on $X$ are **topologically equivalent** if $d$-open sets are the same as $e$-open sets; that is, $\mathscr{T}_d = \mathscr{T}_e$. We omit the proof of the following result, which can be found in MATH21111 or in the literature.

**Proposition 2.3: Lipschitz equivalent metrics are topologically equivalent.**

Suppose that the metrics $d$ and $e$ on a set $X$ are **Lipschitz equivalent**, that is, there are two positive numbers $k$ and $k$ such that

$$\forall x, y \in X, \quad he(x,y) \le d(x,y) \le ke(x,y).$$

Then $d$ and $e$ are topologically equivalent metrics. □

Note that this is **not** an if-and-only-if result: there may be metrics on $X$ which are not Lipschitz equivalent yet are topologically equivalent.

**Remark:** it was shown in MATH21111 *Metric spaces* that the Euclidean metric $d_2$ on $\mathbb{R}^n$ is Lipschitz equivalent to $d_1$ (the "Manhattan metric" or the "taxicab metric") defined by

$$d_1((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|,$$

and also to the metric $d_\infty$, defined by

$$d_\infty((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max_{i=1}^{n} |x_i - y_i|.$$

In fact, for all $x, y \in \mathbb{R}^n$,

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y) \le n d_\infty(x,y),$$

which shows that $d_1$, $d_2$ and $d_\infty$ are pairwise Lipschitz equivalent. By Proposition 2.3, $d_1$, $d_2$ and $d_\infty$ define the same topology on $\mathbb{R}^n$ — the Euclidean topology.

However, Example 2.2 tells us that each of the three metrics defines a **base** for the Euclidean topology; the base consists of open balls of arbitrary radii around each point in the

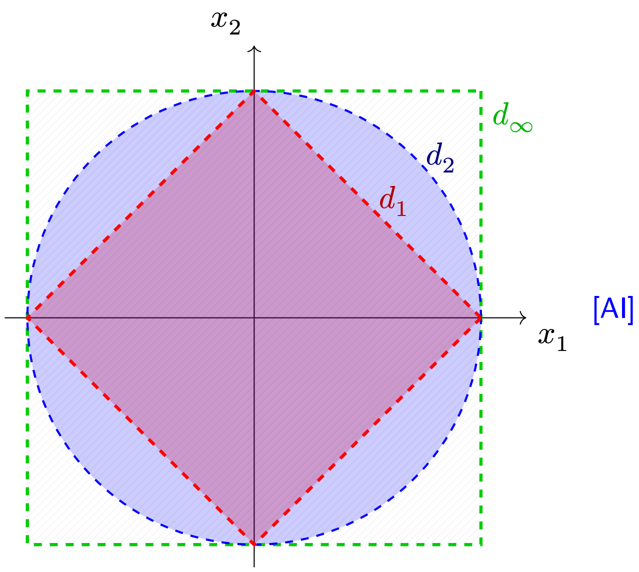



Figure 2.1: *Each of the three metrics $d_1$, $d_2$, $d_\infty$ on $\mathbb{R}^2$ defines its own open unit ball around $(0, 0)$. The three unit balls are shown on the same diagram for ease of comparison.*

plane. We note that these balls are of different shape. Figure 2.1 shows the $d_1$-open ball, the $d_2$-open ball and the $d_\infty$-open ball in $\mathbb{R}^2$, of radius $1$, centred at the same point. We clearly have three different bases for the same Euclidean topology on $\mathbb{R}^2$:

- a base which consists of all open $d_1$-rhombuses around each point;
- a base which consists of all open $d_2$-discs around each point;
- a base which consists of all open $d_\infty$-squares around each point of the plane.

There are, of course, infinitely many more bases for the Euclidean topology on $\mathbb{R}^2$.

## Continuous functions

One of the reasons to introduce a topological space as a more general structure than a metric space is to be able to define continuous functions without a metric.

Let $X$, $Y$ be sets, initially considered without a topological space structure. We write

$f\colon X \to Y$ to denote a function with domain $X$ and codomain $Y$. The words **function, map, mapping** will mean the same thing. The following notation and terminology will be used.

**Definition: image of an element, image of a set, preimage of a set.**

Let $f\colon X \to Y$ be a function. For $x \in X$, the element $f(x)$ of $Y$ is the **image of** $x$ under $f$. For a subset $A \subseteq X$, the subset of $Y$ defined as

$$f(A) = \{f(a) : a \in A\}$$

is the **image of the set** $A$ under $f$. For a subset $B \subseteq Y$, the subset of $X$ defined as

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

is called **the preimage of the set** $B$ under $f$.

We are now going to define **continuity,** which does require a topology on both $X$ and $Y$.

**Definition: continuous function.**

Let $X$ and $Y$ be topological spaces. A function $f\colon X \to Y$ is **continuous** if for every subset $V$ of $Y$, open in $Y$, the preimage $f^{-1}(V)$ is open in $X$.

**Remark:** in MATH21111 *Metric Spaces,* this was shown to be an equivalent definition of continuity. This means that when $X$ and $Y$ are metric spaces, considered with their metric topologies, we can use **two** equivalent definitions of a continuous function:

- the $\varepsilon$ - $\delta$ definition of continuity for functions between metric spaces;
- the topological definition of continuity, given above.

When $X$ or $Y$ is **not** a metric space, we do **not** have the $\varepsilon$ - $\delta$ definition, and can only use the topological definition.

**Warning:** remember,

"$f$ **is continuous**" means $f^{-1}($ **open** $) =$ **open !**

Figure 2.2: *A subset of $\mathbb{R}$ can be open & closed, open, closed, or neither*

It is **not true** for a general continuous function that **images** of open subsets of $X$ are open in $Y$: $f($ open $) \neq$ open !!!

It is often useful to characterise continuous functions in terms of **closed sets,** which we will now define.

## Closed sets

**Definition: closed set.**

Let $X$ be a topological space. A subset $F$ of $X$ is **closed** in $X$ if its complement $X \setminus F$ is open in $X$.

This definition means that every subset $A$ of a topological space $X$ falls into one of the four classes:

- $A$ is open and closed; for example, $A = \emptyset$ or $A = X$;
- $A$ is open but not closed; for example, $X = \mathbb{R}$ (Euclidean topology), $A = (0, 1)$;
- $A$ is closed but not open; for example, $X = \mathbb{R}$ (Euclidean topology), $A = [2, 3]$;
- $A$ is neither open nor closed; for example, $X = \mathbb{R}$ (Euclidean topology), $A = [4, 5)$.

The last three examples are illustrated by Figure 2.2.

**Alert.**

"Not open" does not mean "closed"!

The collection of closed sets in $X$ has properties which mirror, but do not repeat, the properties of open sets – we need to exchange unions and intersections:

**Proposition 2.4: properties of closed sets.**

If $X$ is a topological space, then

(a) $\emptyset$ and $X$ are closed in $X$,

(b) arbitrary intersections of closed sets are closed,

(c) finite unions of closed sets are closed.

*Proof.* Since closed sets are complements of open sets, these properties follow by applying the De Morgan laws to the properties of open sets in Proposition 1.1. □

**Proposition 2.5: the closed set criterion of continuity.**

A function $f\colon X \to Y$ between topological spaces $X$ and $Y$ is continuous if, and only if, the preimage of every closed subset of $Y$ is closed in $X$.

*Proof.* The key point of the proof is the following property of preimages:

$$\forall V \subseteq Y, \quad f^{-1}(Y \setminus V) = X \setminus f^{-1}(V),$$

in other words, **the preimage of a complement is the complement of the preimage.** (See the week 1 tutorial where this was discussed.)

Assume that $f\colon X \to Y$ is continuous. Let $F \subseteq Y$ be any closed set in $Y$. Then $V = Y \setminus F$ is open in $Y$. We compute $f^{-1}(F)$ as follows: $f^{-1}(F) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Since $f^{-1}(V)$ is open (by continuity of $f$), its complement $X \setminus f^{-1}(V)$ is closed, as required. We proved that the preimage of every closed set is closed.

Now assume that the preimage of every closed set under $f$ is closed. Let $V \subseteq Y$ be any open set in $Y$. Then the complement $Y \setminus V$ of $V$ is closed in $Y$, so, by assumption, $f^{-1}(Y \setminus V)$ is closed in $X$. Yet $f^{-1}(Y \setminus V)$ equals $X \setminus f^{-1}(V)$ and, since this set is closed, $f^{-1}(V)$ must be open. We proved that the preimage of every open set is open, and so we have verified the definition of "continuous" for $f$. □

## Easy properties and examples of continuous functions

Unlike in Mathematical Foundations and Analysis, in general we cannot form "sums" or "products" of continuous functions from $X$ to $Y$ because the topological space $Y$ may not have any $+$ or $\times$ operations defined on it. Yet we may form compositions:

**Proposition 2.6: composition of continuous functions is continuous.**

Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ where $X, Y, Z$ are topological spaces, and the functions $f, g$ are continuous. Then the composition $X \xrightarrow{g \circ f} Z$ is also continuous.

*Proof.* The key point of the proof is the formula for the **preimage under composition,** left as an exercise to the student:

$$\forall W \subseteq Z, \quad (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)).$$

Let $W \subseteq Z$ be any open subset of $Z$. Since $g$ is continuous, $g^{-1}(W)$ is open in $Y$. Since $f$ is continuous, $f^{-1}(g^{-1}(W))$ is open in $X$. We have proved that $(g \circ f)^{-1}(W)$ is open in $X$, and so we have verified the definition of "continuous" for $g \circ f$. $\square$

**Example: the identity map is continuous.**

Let $X$ be a topological space. Show that the **identity map on** $X$, $\mathrm{id}_X \colon X \to X$ defined by $x \mapsto x$ for all points $x \in X$, is continuous.

**Solution:** for any $V$ open in $X$, the preimage $\mathrm{id}_X^{-1}(V) = V$ is open. This proves that $\mathrm{id}_X$ is a continuous function.

**Example: a constant function is continuous.**

Let $X, Y$ be topological spaces and let $y_0$ be a point of $Y$. A **constant function** is a function of the form $\mathrm{const}_{y_0} : X \to Y$, $x \mapsto y_0$ for all $x \in X$ (i.e., the function that sends the whole of $X$ to one point). Show that constant functions are continuous.

**Solution:** if $V$ is open in $Y$, the preimage of $V$ under a constant function is as follows:

$$\mathrm{const}_{y_0}^{-1}(V) = \begin{cases} X, & \text{if } y_0 \in V, \\ \emptyset, & \text{if } y_0 \notin V. \end{cases}$$

As $X$ and $\emptyset$ are open in $X$, the preimage of $V$ is always open. Continuity is proved.

## Subspace topology. The inclusion map $\mathrm{in}_A$

Every subset of a topological space is made a topological space in its own right, as follows.

**Definition: subspace topology.**

Let $X$ be a topological space and let $A$ be a subset of $X$.

A subset $V \subseteq A$ is called **open in** $A$ if there exists $U \subseteq X$ such that $U$ is open in $X$ and $V = U \cap A$.

The collection $\mathscr{T}_A$ of subsets of $A$ open in $A$ is a topology on $A$, called the **subspace topology.** By a **subspace** of a topological space $X$ we mean a space $(A, \mathscr{T}_A)$.

The definition of "open in $A$" is illustrated by Figure 2.3. Strictly speaking, we need to prove that a "subspace topology" is indeed a topology. We do not go through the proof, given below, in class, and it is often left as an exercise in the literature.

**Example: subspace topology is indeed a topology.**

Let $X$ be a topological space, and let $A \subseteq X$. Show that $\mathscr{T}_A$ is a topology on $A$.

**Solution** (*not given in class*): Axiom (i) of topology requires $A \in \mathscr{T}_A$. We have $A = X \cap A$ where the set $X$ is open in $X$, hence by definition of "open in $A$", $A \in \mathscr{T}_A$, as required.

Axiom (ii) requires that the union of any subcollection of $\mathscr{T}_A$ be again in $\mathscr{T}_A$. Let $\{V_\alpha : \alpha \in I\}$ be a subcollection of $\mathscr{T}_A$. Then for every $\alpha$, $V_\alpha$ is open in $A$, and so there exists a set $U_\alpha$ open in $X$ such that $V_\alpha = U_\alpha \cap A$. We have

$$\bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I} \Big( U_\alpha \cap A \Big) = \Big( \bigcup_{\alpha \in I} U_\alpha \Big) \cap A$$

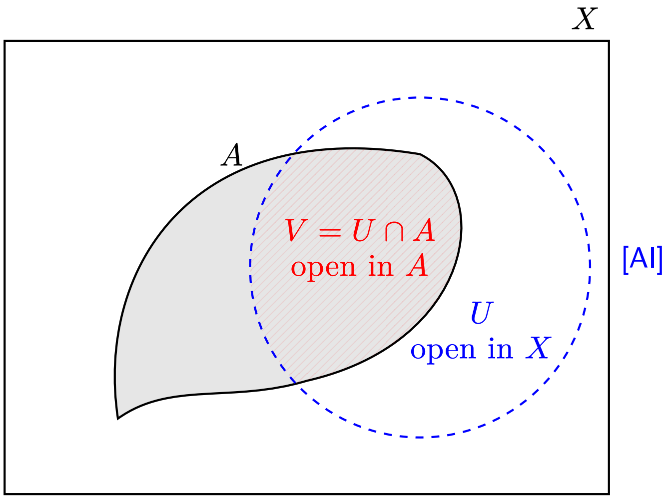



Figure 2.3: *meaning of "a subset $V$ of $A$ is open in $A$".*

(the last step is a known distributive law for $\cap$ and $\cup$). The set $\bigcup_{\alpha\in I} U_\alpha$ is open in $X$ since it is a union of open sets, so by definition of "open in $A$", $\bigcup_{\alpha\in I} V_\alpha \in \mathscr{T}_A$, as required.

Axiom (iii) of topology requires that if $V, V' \in \mathscr{T}_A$ then $V \cap V' \in \mathscr{T}_A$. Let $V, V' \in \mathscr{T}_A$. Then there exist sets $U, U'$, open in $X$, such that $V = U \cap A$ and $V' = U' \cap A$. Then $V \cap V' = (U \cap A) \cap (U' \cap A) = (U \cap U') \cap A$. The set $U \cap U'$ is open in $X$ as an intersection of two open sets, so, by definition of "open in $A$", $V \cap V' \in \mathscr{T}_A$, as required.

To each subspace of $X$ there is associated a continuous map:

**Definition: inclusion map.**

Let $A$ be a subspace of a topological space $X$. The function $\mathrm{in}_A \colon A \to X$, defined by $a \mapsto a$ for all $a \in A$, is the **inclusion map** of $A$.

If $A = X$, we have $\mathrm{in}_X = \mathrm{id}_X$. It turns out that the inclusion map is always continuous:

**Proposition 2.7: the inclusion map is continuous.**

If $A$ is a subspace of a topological space $X$, the inclusion map $\mathrm{in}_A \colon A \to X$ is continuous.

*Proof.* Let $U$ be an arbitrary open subset of $X$. The preimage

$$\mathrm{in}_A^{-1}(U) = \{a \in A : \mathrm{in}_A(a) \in U\} = \{a \in A : a \in U\} = U \cap A$$

is open in $A$ by definition of subspace topology. Continuity of $\mathrm{in}_A$ is proved. □

## References for the week 2 notes

The Euclidean space $\mathbb{R}^n$ is written as $\mathbb{E}^n$ in [Armstrong].

A **base** of a topology is defined in the same way in [Armstrong, Section 2.1] but is called a **basis** in [Sutherland, Definition 8.9].

Proposition 2.3 that **Lipshitz equivalence implies topological equivalence** is [Sutherland, Proposition 6.34]. Metrics $d_1$, $d_2$ and $d_\infty$ were introduced in MATH21111. They are also defined for $n = 2$ in [Sutherland, Example 5.7].

**Figure 2.1** is based on LaTeX/TikZ code generated by OpenAI ChatGPT in response to the following prompt by YB given below. YB made minor edits to the code to improve visual appearance.

> `Can you produce LaTeX or TikZ code which would generate drawing showing, in the same pair of coordinate axes, the image of the` $d_1$`-unit ball around the origin, the` $d_2$`-unit ball around the origin, and the` $d_\infty$`-unit ball around the origin in the plane` $\mathbb{R}^2$`? The three unit balls must be of different color. Here` $d_1$ `denotes the "Manhattan metric" on the plane,` $d_2$ `is the Euclidean metric, and` $d_\infty$ `is the metric where the distance between the points` $(x_1, x_2)$ `and` $(y_1, y_2)$ `is defined as` $\max(|x_1 - x_2|, |y_1 - y_2|)$`.`

The definition of a **continuous function** via preimages of open sets is standard in topology, see [Sutherland, Definition 8.1]. However, [Armstrong] uses a different definition, shown to be equivalent to ours in [Armstrong, Theorem (2.6)]. In this course, we do not need the notion "$f$ is continuous at a point $x$": interested students can check [Sutherland, Definition 8.2].

**Closed sets** are defined in [Sutherland, Definition 9.1], and our Proposition 2.4 is [Sutherland, Proposition 9.4]. Our **closed set criterion of continuity,** Proposition 2.5, is [Sutherland, Proposition 9.5], yet Sutherland omits the proof. Proposition 2.6, **continuity of composition,** is [Sutherland, Proposition 8.4]. Our examples showing that $\mathrm{id}_X$ **is continuous** and **constants are continuous** solve [Sutherland, Exercise 8.1(a,b)].

**Figure 2.3** is based on TikZ code generated by OpenAI ChatGPT when asked to illustrate the definition of subspace topology. The area of $V$ to be shaded was calculated incorrectly by AI, and YB replaced the calculation with a call to the TikZ `clip` function call.

Our definition of **subspace topology** is [Sutherland, Definition 10.3], and the proof that it is a topology solves [Sutherland, Exercise 10.2]. Proposition 2.7, **the inclusion map is continuous,** is [Sutherland, Proposition 10.4].

# Week 2

# Exercises (answers at end)

Version 2024/11/02. To accessible online version of these exercises

**Exercise 2.1.** (a) Prove that the collection $\mathscr{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, +\infty) : x \in \mathbb{R}\}$ is a topology on the set $\mathbb{R}$ of real numbers.

(b) Prove that the collection $\mathscr{N} = \{\emptyset, \mathbb{R}\} \cup \{[x, +\infty) : x \in \mathbb{R}\}$ is not a topology on the set $\mathbb{R}$. Which axiom(s) of topology is/are not satisfied?

**Exercise 2.2.** Consider the set $X = \{1, 2\}$ with two points. Describe all possible topologies $\mathscr{T}$ on $X$. Among the topologies that you describe, identify the discrete topology, the antidiscrete topology and the cofinite topology.

**Exercise 2.3.** Call a subset $A$ of $\mathbb{R}$ "cocountable" if $A = \emptyset$ or $\mathbb{R} \setminus A$ is finite or countably infinite.

(a) Show that the collection of all cocountable subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$.

(b) Is this topology the same as discrete topology? Antidiscrete? Cofinite topology?

# Week 2

# Exercises — solutions

Version 2024/11/02. To accessible online version of these exercises

**Exercise 2.1.** (a) Prove that the collection $\mathscr{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, +\infty) : x \in \mathbb{R}\}$ is a topology on the set $\mathbb{R}$ of real numbers.

(b) Prove that the collection $\mathscr{N} = \{\emptyset, \mathbb{R}\} \cup \{[x, +\infty) : x \in \mathbb{R}\}$ is not a topology on the set $\mathbb{R}$. Which axiom(s) of topology is/are not satisfied?

**Answer to E2.1.** (a) **Axiom (i) of topology** requires that $\mathbb{R}$ is in $\mathscr{T}$. This axiom is satisfied.

**Axiom (ii) of topology** requires that the union, $\bigcup \mathscr{T}_1$, of every subcollection $\mathscr{T}_1$ of $\mathscr{T}$ be a member of $\mathscr{T}$. Let $\mathscr{T}_1$ be a subcollection of $\mathscr{T}$.

If $\mathscr{T}_1$ does not contain non-empty sets, then $\bigcup \mathscr{T}_1 = \emptyset$ which is a member of $\mathscr{T}$.

If $\mathscr{T}_1$ contains $\mathbb{R}$, then $\bigcup \mathscr{T}_1 = \mathbb{R}$ which is a member of $\mathscr{T}$.

The remaining possibility is that $\mathscr{T}_1$ is a family of sets $(x, +\infty)$ where $x$ runs over some subset $I$ of real numbers. Let $m = \inf I$. There are two cases:

- if $m = -\infty$, then $\bigcup \mathscr{T}_1 = \mathbb{R}$ (*justify this!*) which is in $\mathscr{T}$;

- if $m$ is a finite real number, then $\bigcup \mathscr{T}_1 = (m, +\infty)$ (*justify this!*) which is also a member of $\mathscr{T}$.

In all cases, $\bigcup \mathscr{T}_1$ belongs to $\mathscr{T}$, so axiom (ii) is satisfied.

**Axiom (iii) of topology** requires that $A \cap B \in \mathscr{T}$ for any two sets $A, B \in \mathscr{T}$. Let $A, B \in \mathscr{T}$. The following case-by-case analysis shows that $A \cap B \in \mathscr{T}$:

| $A, B$ | $A \cap B$ | $\in \mathscr{T}$? |
|---|---|---|
| $A = \emptyset$ or $B = \emptyset$ | $\emptyset$ | Yes |
| $A = \mathbb{R}$ | $B$ | Yes |
| $B = \mathbb{R}$ | $A$ | Yes |
| $A = (x, +\infty), B = (y, +\infty)$ | $(\max(x, y), +\infty)$ | Yes |

(b) One can see that axioms (i) and (iii) are satisfied, but $\mathscr{N}$ fails axiom (ii). Indeed, consider the following infinite union of sets from $\mathscr{N}$:

$$\bigcup_{x>0} [x, +\infty) = (0, +\infty).$$

This union is not a set in $\mathscr{N}$. We have proved that $\mathscr{N}$ is not a topology.

**Exercise 2.2.** Consider the set $X = \{1, 2\}$ with two points. Describe all possible topologies $\mathscr{T}$ on $X$. Among the topologies that you describe, identify the discrete topology, the antidiscrete topology and the cofinite topology.

**Answer to E2.2.** There are four subsets of $X$: $\emptyset$, $\{1\}$, $\{2\}$ and $X$.

Let $\mathscr{T}$ be a topology. Axiom (i) of topology requires $X \in \mathscr{T}$, and Proposition 1.1 tells us that $\emptyset \in T$. This leaves us with four options, because we can either include or exclude each of the two *singleton sets*, $\{1\}$ and $\{2\}$. It is easy to see that **all four** collections are topologies on $X$:

- $\{\emptyset, X\}$: the antidiscrete topology.
- $\{\emptyset, \{1\}, X\}$.
- $\{\emptyset, \{2\}, X\}$.

- $\{\emptyset, \{1\}, \{2\}, X\}$: the collection of all subsets of $X$, that is, the discrete topology.

Note that for a finite set $X$, the discrete topology on $X$ and the cofinite topology on $X$ are the same thing. Indeed, in the cofinite topology, open sets are $\emptyset$ and all subsets of $X$ with finite complement. Yet when $X$ is finite, every subset of $X$ has finite complement. Hence, for finite $X$, all subsets of $X$ are open in cofinite topology, and all subsets of $X$ are open in the discrete topology.

**Exercise 2.3.** Call a subset $A$ of $\mathbb{R}$ "cocountable" if $A = \emptyset$ or $\mathbb{R} \setminus A$ is finite or countably infinite.

(a) Show that the collection of all cocountable subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$.

(b) Is this topology the same as discrete topology? Antidiscrete? Cofinite topology?

**Answer to E2.3.** (a) We modify the proof of Proposition 1.4 which deals with cofinite sets. We will change "finite" to "countable". By "countable" we mean "finite or countably infinite".

We need to prove that the collection $\mathscr{C}$ which consists of the empty set and all subsets of $X$ with countable complement is a topology on the set $X$. Let us show that $\mathscr{C}$ fulfils axioms (i)–(iii) from the definition of topology.

(i) $X$ has complement $\emptyset$, and $\emptyset$ is countable, so $X \in \mathscr{C}$.

(ii) Let $\mathscr{F}$ be some collection of sets from $\mathscr{C}$. If all sets in $\mathscr{F}$ are empty, then $\bigcup \mathscr{F} = \emptyset \in \mathscr{C}$.

Otherwise, take a non-empty set $U \in \mathscr{F}$. Then $U$ must have countable complement, and $U \subseteq \bigcup \mathscr{F}$, so by lemma 1.2, $X \setminus \bigcup \mathscr{F} \subseteq X \setminus U$. Yet $X \setminus U$ is a countable set, and all subsets of a countable set are countable. Hence the complement of $\bigcup \mathscr{F}$ is countable, proving that $\bigcup \mathscr{F}$ is in $\mathscr{C}$.

(iii) Suppose $U, V \in \mathscr{C}$. If one of $U, V$ is an empty set, then $U \cap V = \emptyset \in \mathscr{C}$.

Otherwise, $U$ and $V$ are non-empty, and since they are in $\mathscr{C}$, $U$ and $V$ must have countable complements. Then by the De Morgan laws $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$. Thus, $U \cap V$ has a countable complement (a union of two countable sets) and so $U \cap V \in \mathscr{C}$.

(b) We show that the cocountable topology on $\mathbb{R}$ is not the same as the discrete, the antidiscrete and the cofinite topologies. To show that two topologies are different, we exhibit a subset of $\mathbb{R}$ which is open in one topology but is not open in the other topology.

- the set $\mathbb{N}$ is not open in the cocountable topology ($\mathbb{R} \setminus \mathbb{N}$ is uncountable), yet $\mathbb{N}$ is open in the discrete topology (all sets are open in the discrete topology)
- the set $\mathbb{R} \setminus \mathbb{N}$ is open in the cocountable topology because its complement $\mathbb{N}$ is countable, yet $\mathbb{R} \setminus \mathbb{N}$ is not open in the antidiscrete topology (it is not one of $\emptyset$, $\mathbb{R}$) and is not open in the cofinite topology (its complement $\mathbb{N}$ is not finite).

## References for the exercise sheet

The answer to E2.1(a) essentially solves [Sutherland, Exercise 7.6].

E2.2 is [Sutherland, Exercise 7.1a], and classification of topologies on $\{1, 2\}$ answers the second part of [Sutherland, Exercise 7.5].

# Week 3

# Homeomorphic spaces. Topological properties. Hausdorff spaces

Version 2024/11/02 To accessible online version of this chapter

## Homeomorphisms and homeomorphic spaces

We have arrived at one of the most important definitions of the course.

**Definition: homeomorphism, homeomorphic spaces.**

Let $X$ and $Y$ be topological spaces. A function $f\colon X \to Y$ is a **homeomorphism** if

- $f$ is a bijection,
- $f$ is continuous,
- the inverse, $f^{-1}\colon Y \to X$, of $f$, is continuous.

Two topological spaces $X, Y$ are **homeomorphic** if there exists a homeomorphism $f\colon X \to Y$.

Note that the topologies of two homeomorphic spaces are considered equivalent. Indeed, if $f$ is a homeomorphism between the topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$, then there is a one-to-one correspondence

$$\mathscr{T}_X \to \mathscr{T}_Y, \quad U \mapsto f(U)$$

between the two topologies. Indeed, if $U \subseteq X$ is open then $f(U) = (f^{-1})^{-1}(U)$ is also open, because $f^{-1}$ is continuous by definition of a homeomorphism; and every open set $V \subseteq Y$ is an image, under $f$, of exactly one open set, $f^{-1}(V)$, in $X$.

> **Notation: homeomorphism.**
>
> We will write $X \xrightarrow{\sim} Y$ to indicate that there is a homeomorphism between $X$ and $Y$, or $X \underset{f}{\xrightarrow{\sim}} Y$ to indicate that $f\colon X \to Y$ is a homeomorphism.

Earlier, we used the word "equivalent" to describe a relation between topological spaces which are homeomorphic. This word has a precise mathematical meaning:

> **Claim: 'homeomorphic' is an equivalence relation.**
>
> 'Homeomorphic' is an equivalence relation between topological spaces.

*Proof.* We need to check the three conditions of equivalence relation.

**Reflexive:** we need to show that every space $X$ is homeomorphic to itself. Indeed, $X \underset{\mathrm{id}_X}{\xrightarrow{\sim}} X$. The identity map $\mathrm{id}_X$ is a bijection, is continuous as shown earlier, and $\mathrm{id}_X^{-1} = \mathrm{id}_X$ hence the inverse is also continuous.

**Symmetric:** We need to show that if $X \xrightarrow{\sim} Y$, then $Y \xrightarrow{\sim} X$. Assume that $X \underset{f}{\xrightarrow{\sim}} Y$. Since $f$ is a homeomorphism, the inverse bijection $f^{-1}$ is, by definition, continuous. Furthermore, the inverse of $f^{-1}$ is $f$ which is continuous. We conclude that $Y \underset{f^{-1}}{\xrightarrow{\sim}} X$.

**Transitive:** we need to show that if $X \xrightarrow{\sim} Y$ and $Y \xrightarrow{\sim} Z$, then $X \xrightarrow{\sim} Z$. Assume that $X \underset{f}{\xrightarrow{\sim}} Y \underset{g}{\xrightarrow{\sim}} Z$. Then $X \underset{g \circ f}{\xrightarrow{\sim}} Z$ where $g \circ f$ is a homeomorphism: it is a bijection as it has inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, and both $g \circ f$ and $f^{-1} \circ g^{-1}$ are continuous as compositions of continuous maps, by Proposition 2.6. □

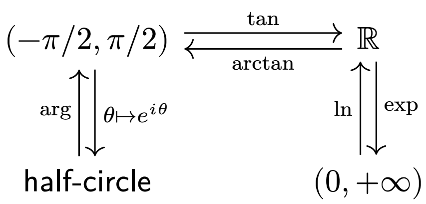
Let us now consider examples of homeomorphic spaces.

**Example: some spaces homeomorphic to $\mathbb{R}$.**

Show that the Euclidean line $\mathbb{R}$ is homeomorphic to

- the open interval $(-\pi/2, \pi/2)$ (a subspace of $\mathbb{R}$);
- an open half-line $(0, +\infty)$;
- the right open half-circle of the unit circle in the complex plane.

**Solution.** We construct pairs of continuous functions which are mutual inverses between some pairs of the given spaces:




We do not need to construct homeomorphisms between **each** pair of the given spaces: since "homeomorphic" is an equivalence relation, we have constructed enough to show that all four spaces are homeomorphic to each other.

The homeomorphism between the right half-circle of the unit circle and $\mathbb{R}$ can be described in purely geometric terms as in Figure 3.1: take a point of the half-circle and project it onto the vertical tangent line to the half-circle at the point $(1, 0)$ in the plane.

The homeomorphisms shown above are not unique, and there are many other ways of showing that these four spaces are pairwise homeomorphic.

**Remark.** The four spaces shown above are quite different as **metric spaces:** for example,

- $\mathbb{R}$ and $(0, +\infty)$ are unbounded but $(-\pi/2, \pi/2)$ and the half-circle are bounded;
- $\mathbb{R}$ is a complete metric space whereas $(0, +\infty)$, $(-\pi/2, \pi/2)$ and the open half-circle are not complete.

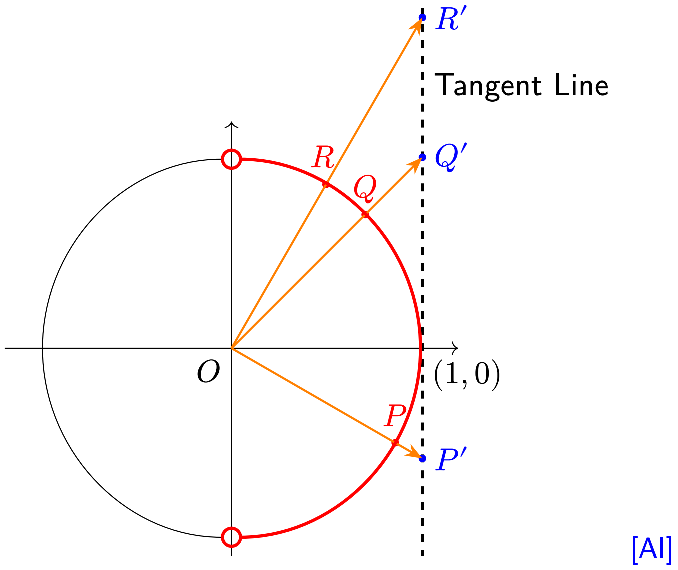



Figure 3.1: *a homeomorphism between the open half-circle and a straight line*

Hence there is **no way** these metric spaces can be **isometric** (i.e., equivalent as metric spaces). We thus observe that being **homeomorphic** (i.e., equivalent as topological spaces) is a weaker requirement, for metric spaces, than being isometric.

**Example: some spaces homeomorphic to $\mathbb{R}^2$.**

Show that the Euclidean plane $\mathbb{R}^2$ is homeomorphic to

- the **punctured sphere** (a sphere in $\mathbb{R}^3$ with one point removed);
- the open unit disc $B_1((0,0))$ in $\mathbb{R}^2$;
- the open quadrant $\{(x, y) \in \mathbb{R}^2 : x, y > 0\}$.

**Solution:** we exhibit a homeomorphism between $\mathbb{R}^2$ and a punctured sphere, leaving the other spaces to the student.

Consider the unit sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$, let $N$ be the point $(0, 0, 1)$ — "the North Pole" of the sphere — and let $S$ be the point $(0, 0, -1)$, "the South pole". We construct

$$f \colon S^2 \setminus \{N\} \to \mathbb{R}^2,$$

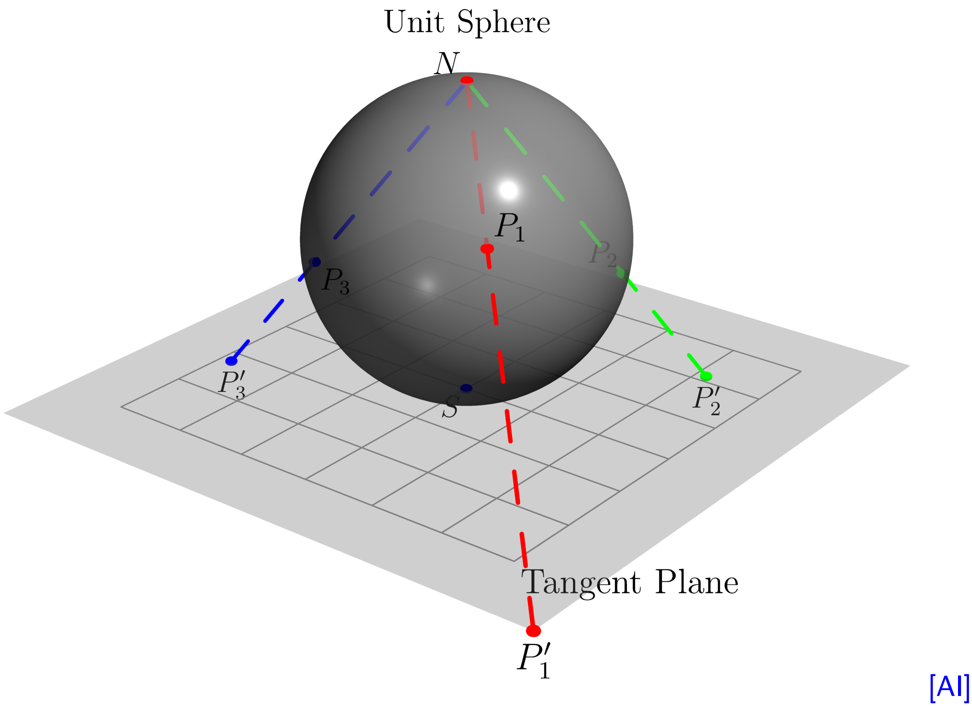



Figure 3.2: *the stereographic projection is a homeomorphism between a punctured sphere and* $\mathbb{R}^2$. [Link to online interactive 3D diagram]

where $\mathbb{R}^2$ is identified with the plane $\{z = -1\}$ tangent to the sphere at $S$. The function $f$ is defined as follows:

- let $P$ be a point on $S^2 \setminus \{N\}$;
- extend the straight line $NP$ beyond $P$;
- let $P'$ be the point of intersection of the line $NP$ with the plane $\{z = -1\}$;
- put $f(P) = P'$.

The construction of $f$ is illustrated by the interactive Figure 3.2. The map $f$ is known as the **stereographic projection.** Homeomorphisms between subsets of the sphere and subsets of a plane have practical applications in **cartography** (the science of drawing maps).

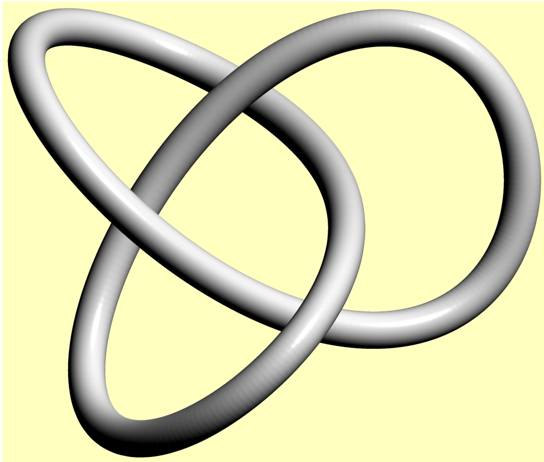


Figure 3.3: *a trefoil knot* [Link to online interactive 3D diagram]

## Further examples and applications of homeomorphisms

*The material in this section is not examinable*

**Knot theory** is a branch of mathematics which arose from an attempt to classify knots rigorously by Peter Guthrie Tait (1831–1901). In 1920s, following the work of J. Alexander, knot theory became part of Topology.

A **knot** is a smooth injective function $S^1 \to \mathbb{R}^3$ where $S^1$ is the unit circle. The image of the function is a smooth closed curve in $\mathbb{R}^3$ which has no self-intersections; such curves are themselves called knots. Two knots $K_1$, $K_2$ are equivalent, or **isotopic,** if there is a continuous deformation of the space $\mathbb{R}^3$ which transforms $K_1$ into $K_2$.

The main problem of knot theory is to classify all knots up to isotopy. For example, a basic result of knot theory is that a **trefoil knot,** the curve shown in Figure 3.3, is not isotopic to the **unknot** (a straightforward copy of the circle in $\mathbb{R}^3$).

Is the problem of deciding whether two knots are isotopic related to the notion of homeomorphism? After all, every knot is homeomorphic to a circle, isn't it? Yes, but **complements** of knots are not homeomorphic. A highly non-obvious result is

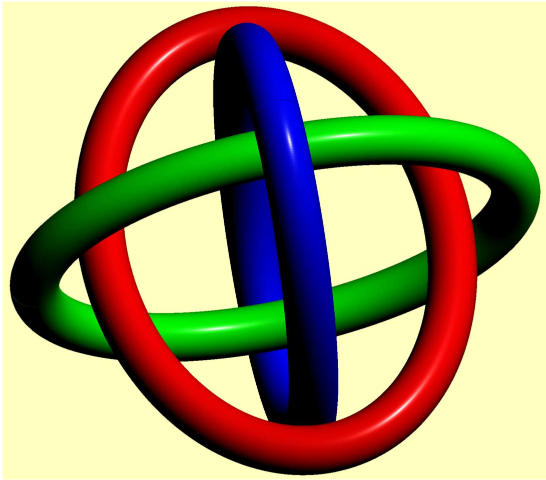
 [AI]

Figure 3.4: *Borromean rings* [Link to online interactive 3D diagram]

**Theorem: the Gordon-Luecke theorem (1989).**

Two knots $K_1$, $K_2$ in $\mathbb{R}^3$ are isotopic (up to taking a mirror image) if, and only if, there exists a homeomorphism $\mathbb{R}^3 \setminus K_1 \xrightarrow{\sim} \mathbb{R}^3 \setminus K_2$.

The theory of knots extends to **links** which are unions of disjoint knots in $\mathbb{R}^3$. An example of a non-trivial link is the famous **Borromean rings:** three interlinked circles in $\mathbb{R}^3$, such that if any one circle is removed, the remaining two circles become unlinked, see Figure 3.4. Unlike knots, a link is not determined up to isotopy by a homeomorphism class of its complement.

A stylised representation of Borromean rings was chosen as the logo of the International Mathematical Union.

*End of non-examinable material.*

## Topological properties

The following general type of a topological problem is of overwhelming importance in theory as well as applications.

**Problem 1: the homeomorphism problem.**

Given two topological spaces $X$, $Y$, determine whether $X$ and $Y$ are homeomorphic.

A homeomorphism $X \xrightarrow{\sim} Y$, if exists, can often be constructed explicitly as a map. Yet it may not be obvious how to justify a negative answer to Problem 1. The following concept is useful as it allows us to prove, in many cases, that two topological spaces are **not** homeomorphic.

**Definition: topological property.**

A property of topological spaces is called a **topological property** if, whenever a space $X$ has this property, all spaces homeomorphic to $X$ also have this property.

A standard approach to proving that two topological spaces $X$, $Y$ are **not** homeomorphic is to find a topological property of $X$ which is not shared by $Y$.

In simple cases, this can be achieved by going through a list of well-known topological properties and determining which of them $X$ and $Y$ have. We will now start putting together a (short) list of the most fundamental and important topological properties.

Among topological properties, we distinguish those which help us to solve the second main problem:

**Problem 2: the continuous image problem.**

Given two topological spaces $X$, $Y$, determine whether there exists a continuous surjective map $X \to Y$.

Of course, a negative answer to Problem 2 implies a negative answer to Problem 1. Problem 2 can be solved with the help of a topological property of $X$ which must be shared by all continuous images of $X$ and not just spaces homeomorphic to $X$. We will analyse each topological property to see if it helps us to solve Problem 2.

## Hausdorff spaces

The first property of a topological space that we consider is being a Hausdorff space.

**Definition: Hausdorff space.**

A topological space $X$ is **Hausdorff** if

$$\forall x, y \in X,\ x \neq y \implies \exists U, V \text{ open in } X\text{: } x \in U,\ y \in V,\ U \cap V = \emptyset.$$

In other words, **two distinct points of $X$ must have disjoint open neighbourhoods.**

**Proposition 3.1.**

The Hausdorff property is a topological property.

*Proof.* Given topological spaces $X$, $Y$ such that $X$ is Hausdorff and $Y \underset{f}{\overset{\sim}{\to}} X$, we need to prove that $Y$ is Hausdorff.

Let $a, b$ be points of $Y$ with $a \neq b$. The points $f(a), f(b)$ of $X$ are distinct as $f$ is injective. Hence, applying the definition of "Hausdorff" to $X$, we can find

- $V_{f(a)}$, $V_{f(b)}$ open in $X$ such that $f(a) \in V_{f(a)}$, $f(b) \in V_{f(b)}$, $V_{f(a)} \cap V_{f(b)} = \emptyset$.

Put $U_a = f^{-1}(V_{f(a)})$ and $U_b = f^{-1}(V_{f(b)})$. Observe that

- $U_a$, $U_b$ are open in $Y$ because $f$ is continuous so $f^{-1}(\text{open}) = \text{open}$;
- $a \in U_a$, $b \in U_b$;
- $U_a \cap U_b = f^{-1}(V_{f(a)} \cap V_{f(b)}) = f^{-1}(\emptyset) = \emptyset$.

We have thus verified the definition of "Hausdorff" for the space $Y$. □

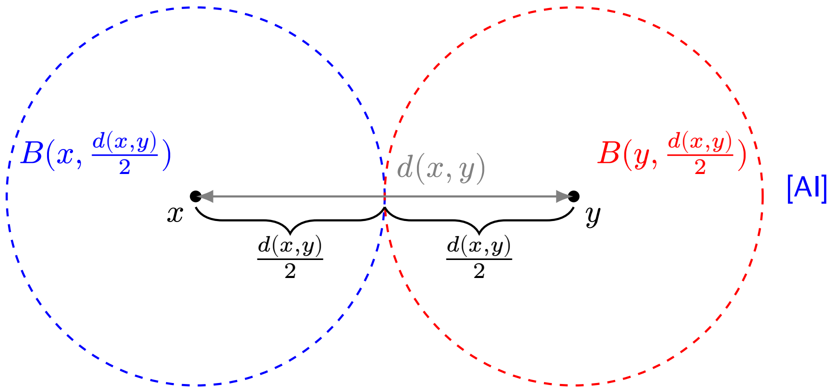



Figure 3.5: *"Metric topology is Hausdorff"*

Many "natural" examples of topological spaces are Hausdorff, for the following reason:

**Proposition 3.2.**

Metric topology is Hausdorff.

*Proof.* Let $(X, d)$ be a metric space. Let $x, y$ be points of $X$ such that $x \neq y$; then, by axioms of a metric, the distance $d(x, y)$ is positive. Denote $r = \frac{d(x,y)}{2}$ and consider the open balls $U = B_r(x)$ and $V = B_r(y)$.

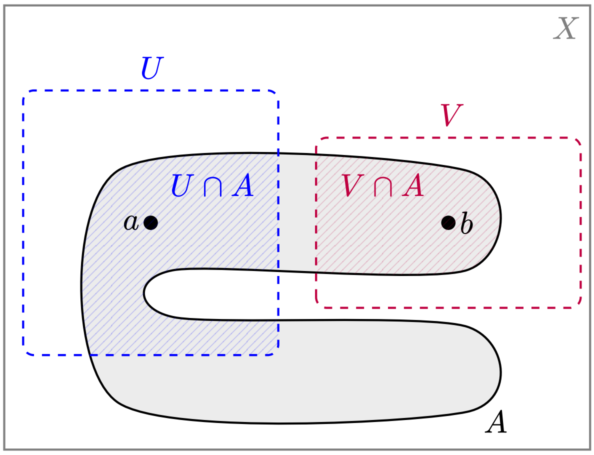
Then $U \ni x$, $V \ni y$, and a standard argument based on the triangle inequality shows that $U \cap V = \emptyset$. (See Figure 3.5 for an illustration.)

We have constructed disjoint open neighbourhoods of $x, y$ and so we have verified the definition of "Hausdorff" for $X$. $\square$

**Proposition 3.3.**

A subspace of a Hausdorff space is Hausdorff.

*Proof.* Let $X$ be a Hausdorff topological space, and let $A$ be a subset of $X$ considered with the subspace topology.




[AI]

Figure 3.6: *"a subspace of a Hausdorff space is Hausdorff"*

Take two distinct points $a, b$ of $A$. Then $a, b$ are also distinct points of $X$, and so they have disjoint open neighbourhoods, $U \ni a$ and $V \ni b$, in $X$.

The sets $U$ and $V$ may not be subsets of $A$, so we put $U' = U \cap A$ and $V' = V \cap A$. Then

- $U' \ni a$ and $V' \ni b$;
- $U'$, $V'$ are open in $A$ by definition of the subspace topology;
- $U' \cap V' = (U \cap A) \cap (V \cap A) = (U \cap V) \cap A = \emptyset \cap A = \emptyset$.

We have constructed disjoint open neighbourhoods of $a, b$ in $A$ and so we have verified the definition of "Hausdorff" for $A$. (See Figure 3.6 for an illustration.) □

**Proposition 3.4.**

In a Hausdorff space, a point is closed.

*Proof.* **Attention:** strictly speaking, a point is not a set and so cannot be closed. Still, "*a point is closed*" is a traditional shorthand for saying "*a set which consists of a single point is closed*", or, equivalently, "*a singleton is closed*": *singleton* means a one-point set.

Let $x \in X$ where $X$ is a Hausdorff topological space. We need to show that the set $\{x\}$ is closed, equivalently that its complement $X \setminus \{x\}$ is open in $X$.

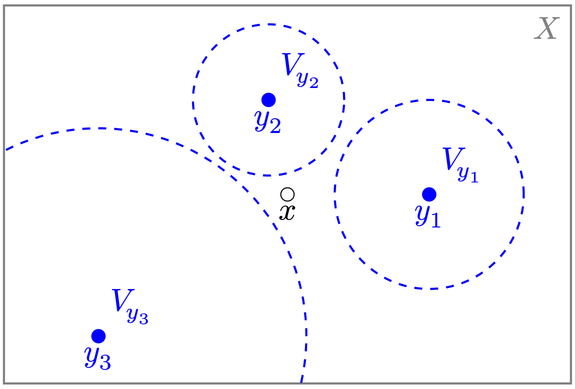



Figure 3.7: *the set $X \setminus \{x\}$ is the union of the open neighbourhoods $V_y$ for all $y \neq x$*

For each point $y \in X \setminus \{x\}$, we have $y \neq x$ so by definition of "Hausdorff", there are open neighbourhoods $U_y \ni x$, $V_y \ni y$ which are disjoint: $U_y \cap V_y = \emptyset$. (Note that the set $U_y$ depends on $y$, hence we subscript it with $y$ even though it is an open neighbourhood of the point $x$.)

We are going to ignore $U_y(x)$ and only use the fact that $V_y$ does not contain $x$. That is, $y \in V_y \subseteq X \setminus \{x\}$.

The set $\bigcup_{y \in X \setminus \{x\}} V_y$

- is open as a union of open sets $V_y$;
- is contained in $X \setminus \{x\}$, because $V_y \subseteq X \setminus \{x\}$ for each $y$;
- contains $X \setminus \{x\}$, because each point $y$ of $X \setminus \{x\}$ is contained in the set $V_y$.

We conclude that $\bigcup_{y \in X \setminus \{x\}} V_y$ is, in fact, equal to $X \setminus \{x\}$. (See the illustration in Figure 3.7.) Therefore, we have proved that $X \setminus \{x\}$ is open. □

Do non-Hausdorff topological spaces exist? Yes, and they form an important class of topological spaces used in algebraic geometry (search: *Zariski topology*). We give a very simple example of a non-Hausdorff space.

**Example: a non-Hausdorff topological space.**

Show that the set $X = \{1, 2\}$ with the antidiscrete topology is not Hausdorff.

**Solution:** the only open sets in $X$ are $\emptyset$ and $\{1, 2\}$, so the only open neighbourhood of the point $1$ is $\{1, 2\}$. Also, the only only open neighbourhood of the point $2$ is $\{1, 2\}$. Hence the distinct points $1$ and $2$ do not have disjoint open neighbourhoods, showing that $X$ is not Hausdorff.

We note informally that the antidiscrete topology is the "weakest possible" topology as it has the fewest open sets, and there are not enough open sets to provide disjoint open neighbourhoods for points of the space. We will now formalise the notion of weaker (and stronger) topology.

**Definition: stronger topology, weaker topology.**

Suppose that $\mathscr{T}$ and $\mathscr{T}'$ are two topologies on the same set $X$. We say that the topology $\mathscr{T}'$ is **stronger** than $\mathscr{T}$ if every set, open in $\mathscr{T}$, is also open in $\mathscr{T}'$.

We say that $\mathscr{T}$ is **weaker** than $\mathscr{T}'$ if $\mathscr{T}'$ is stronger than $\mathscr{T}$. In summary,

$$\mathscr{T} \subseteq \mathscr{T}' \text{ means that } \mathscr{T} \text{ is weaker, } \mathscr{T}' \text{ is stronger.}$$

**Exercise:** let $X$ be an arbitrary set.

1. Show that any topology on $X$ is stronger than the antidiscrete topology on $X$ and is weaker than the discrete topology on $X$.
2. Show that ($X$, discrete topology) is Hausdorff.
3. Show that ($X$, antidiscrete topology) is non-Hausdorff if, and only if, the cardinality of $X$ is at least $2$.

**Proposition 3.5.**

If $(X, \mathscr{T})$ is a Hausdorff topological space, and a topology $\mathscr{T}'$ on $X$ is stronger than $\mathscr{T}$, then $(X, \mathscr{T}')$ is also Hausdorff.

*Proof.* Take any two points $x, y$ in $X$ such that $x \neq y$. Since $\mathscr{T}$ is Hausdorff, there exist

$\mathscr{T}$-open sets $U \ni x$, $V \ni y$ with $U \cap V = \emptyset$.

Since $\mathscr{T}'$ is stronger than $\mathscr{T}$, the sets $U$, $V$ are open in $\mathscr{T}'$ as well. We have found disjoint $\mathscr{T}'$-open neighbourhoods of $x, y$ and so we have verified the definition of "Hausdorff" for $\mathscr{T}'$. □

## References for the week 3 notes

Definition of **homeomorphism** is the same as [Sutherland, Definition 8.7]. Our claim that **'homeomorphic' is an equivalence relation** solves [Sutherland, Exercise 8.4].

**Figure 3.1** is based on TikZ code written by OpenAI ChatGPT when asked to illustrate the homeomorphism between right open half-circle of the unit circle in the plane and the vertical straight line tangent to the half-circle at the point $(1, 0)$. The code improved by the AI as a result of feedback from YB on incorrect attempts (e.g., "some points that you are using are not on the right half-circle") and underwent minor visual re-styling by YB.

The **stereographic projection** $S^2 \setminus \{N\} \to \mathbb{R}^2$ is defined in [Armstrong, Figure 1.24]. **Figure 3.2** illustrating the stereographic projection is produced by code written by OpenAI ChatGPT in the Asymptote language intended for 3D technical drawing. Two earlier attempts were corrected by AI based on verbal feedback by YB.

The **Gordon-Luecke Theorem** (non-examinable) was proved in: C. M. A. Gordon and J. Luecke, Knots are determined by their complements, *J. Amer. Math. Soc.* **2** (1989), no. 2, 371–415.

The advanced subject of **knot theory** is touched upon in [Armstrong, Chapter 10]. **Figure 3.4,** showing the **link** known as **the Borromean rings** is produced by code written by OpenAI ChatGPT in the Asymptote language. Several incorrect diagrams were generated when the AI tried to represent the rings by perfect circles which is impossible. This diagram, which uses ellipses, was the response to feedback that the previous attempt was too visually complex.

We use the term **topological property** in the same sense as [Armstrong] and [Sutherland], but these textbooks do not formally define this term.

**Figure 3.5** is TikZ code written by OpenAI ChatGPT to a prompt "illustrate the proof that metric topology is Hausdorff".

**Figure 3.6** is based on TikZ code by OpenAI ChatGPT to a prompt "illustrate the proof that a subspace of a Hausdorff space is Hausdorff". Reworked by YB to add visual sophistication.

Proposition 3.2 (**metric implies Hausdorff**) is [Sutherland, Proposition 11.5].
Proposition 3.1 (**Hausdorffness is a topological property**) and Proposition 3.3 (**subspace of Hausdorff is Hausdorff**) solve [Sutherland, Exercise 11.4a,d].
Proposition 3.4 (**in Hausdorff, a point is closed**) solves [Sutherland, Exercise 11.2a].
The example **"antidiscrete $\{1, 2\}$ is not Hausdorff"** answers [Sutherland, Exercise 11.1].

[Sutherland, Definition 7.6] says "finer" and "coarser" in place of our **stronger** and **weaker** topology; we follow the terminology in [Willard] which is also used in Functional Analysis. Proposition 3.5 (**topology stronger than Hausdorff is Hausdorff**) elaborates on the remark made in [Willard, Example 13A.2].

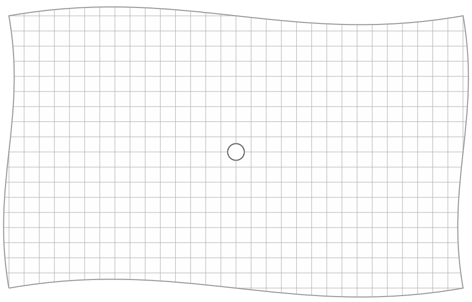
# Week 3

# Exercises (answers at end)

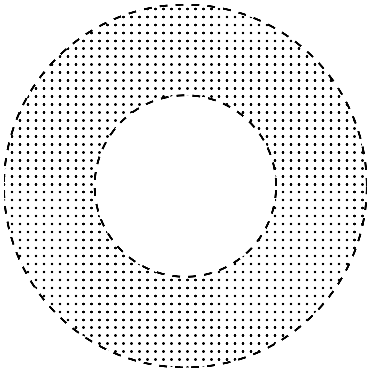
Version 2024/11/10. To accessible online version of these exercises

**Exercise 3.1.** Here is the unseen exercise done in the week 03 tutorial.

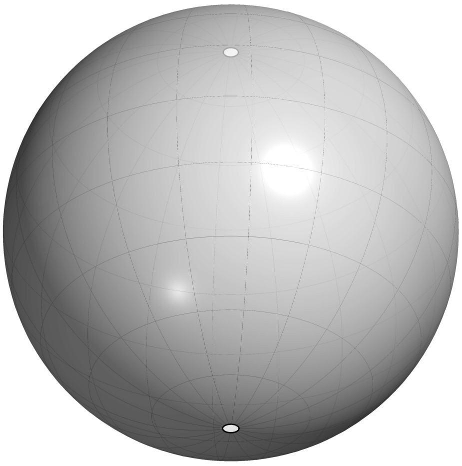
Consider the following topological spaces (the first five are viewed as subspaces of the Euclidean space). Determine which of these spaces are homeomorphic. Give a convincing description, or an explicit formula, for the homeomorphism where necessary; give reasons when the spaces are not homeomorphic.



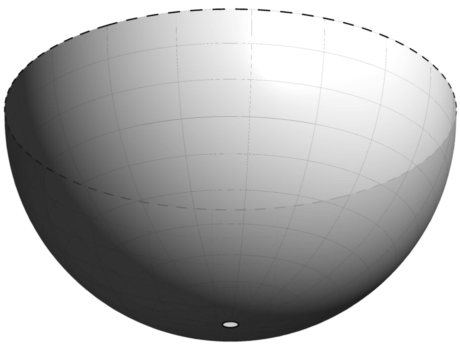
1. The punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$



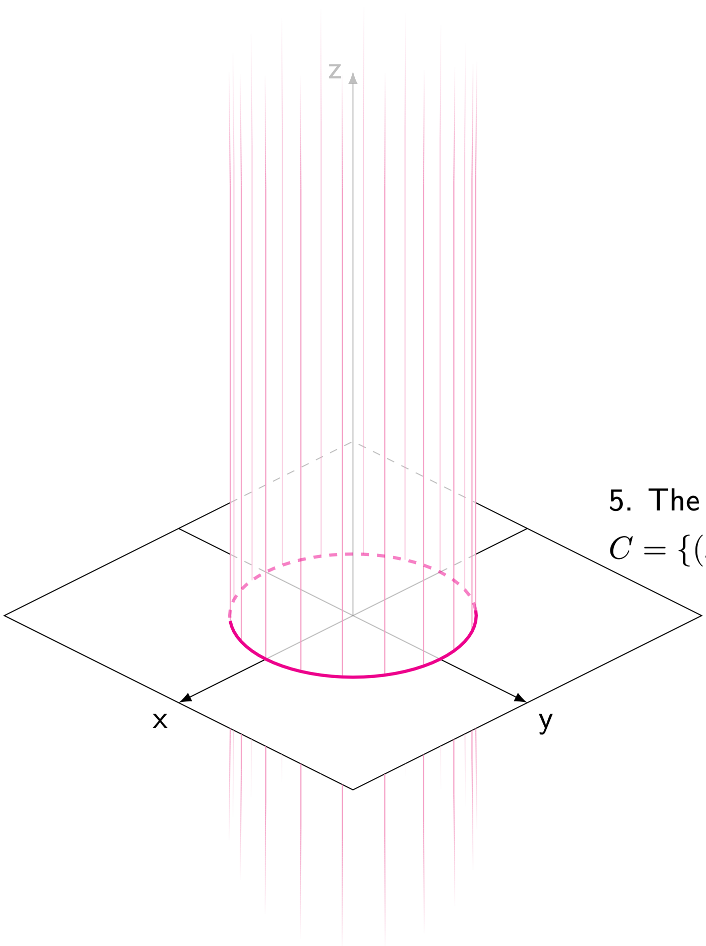
2. The open annulus
$A = \{(x, y) \in \mathbb{R}^2 : 1 < \sqrt{x^2 + y^2} < 2\}$



3. The twice-punctured sphere
$S^2 \setminus \{N, S\}$, where
$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$,
$N = (0, 0, 1)$ and $S = (0, 0, -1)$



4. The punctured open hemisphere
$(S^2 \cap \{z < 0\}) \setminus \{S\}$




5. The (infinite) cylinder
$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$

6. and finally, the set $\mathbb{R}^2$ with antidiscrete topology.

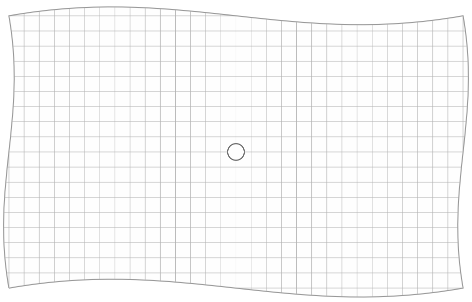
# Week 3

# Exercises — solutions

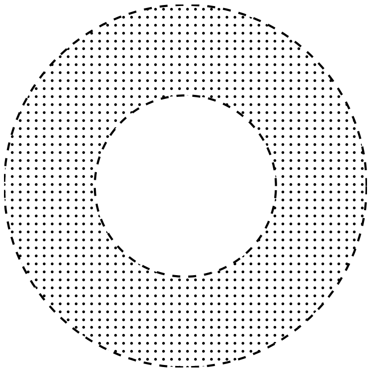
Version 2024/11/10. To accessible online version of these exercises

**Exercise 3.1.** Here is the unseen exercise done in the week 03 tutorial.

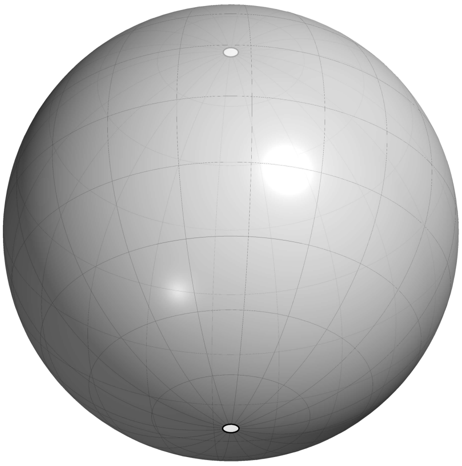
Consider the following topological spaces (the first five are viewed as subspaces of the Euclidean space). Determine which of these spaces are homeomorphic. Give a convincing description, or an explicit formula, for the homeomorphism where necessary; give reasons when the spaces are not homeomorphic.



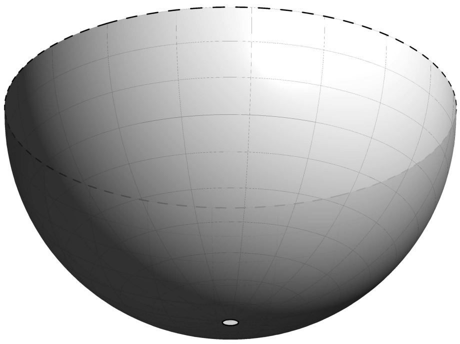
1. The punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$



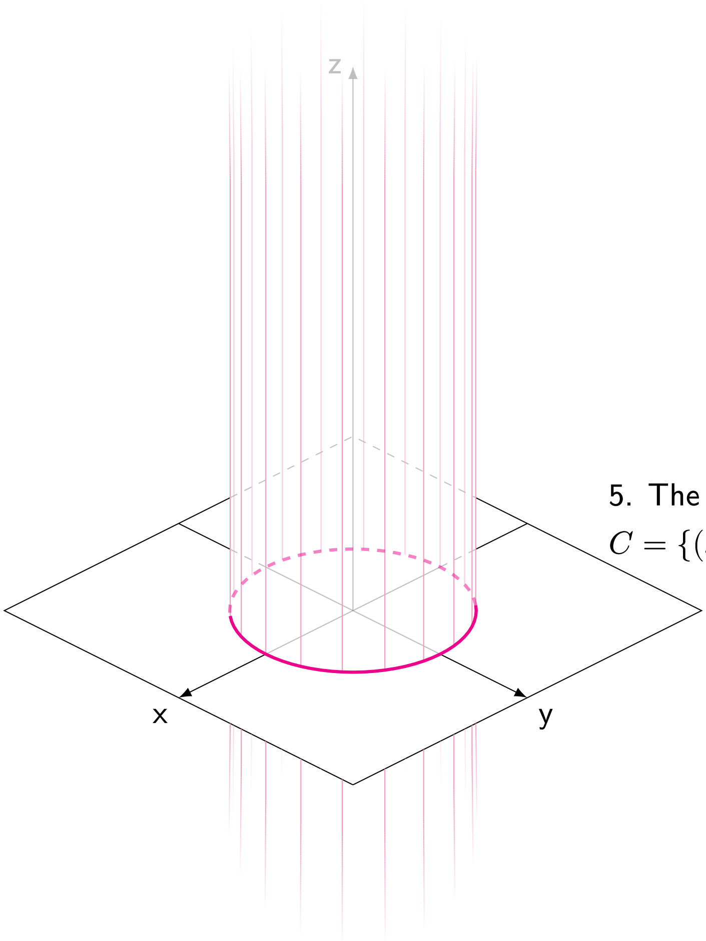
2. The open annulus
$A = \{(x, y) \in \mathbb{R}^2 : 1 < \sqrt{x^2 + y^2} < 2\}$



3. The twice-punctured sphere
$S^2 \setminus \{N, S\}$, where
$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$,
$N = (0, 0, 1)$ and $S = (0, 0, -1)$



4. The punctured open hemisphere
$(S^2 \cap \{z < 0\}) \setminus \{S\}$



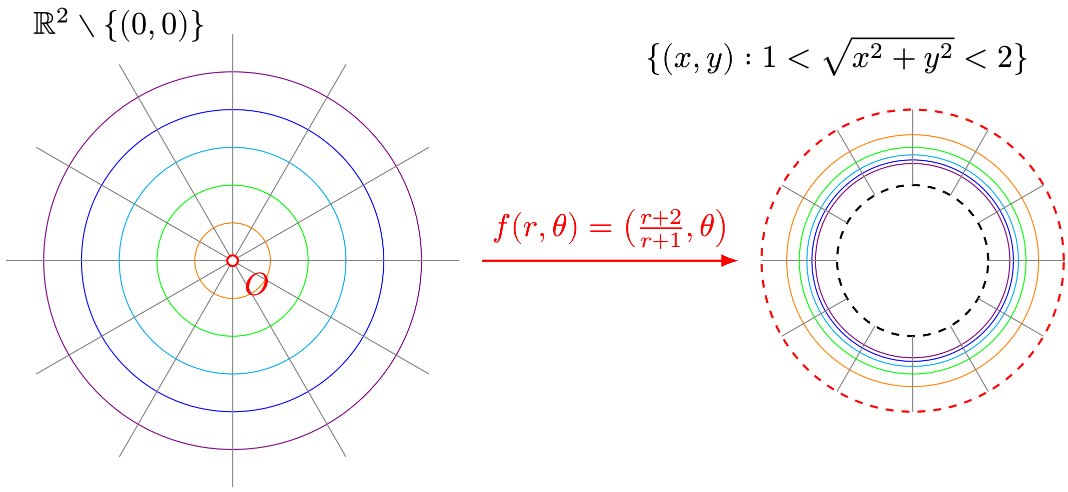

5. The (infinite) cylinder
$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$

6. and finally, the set $\mathbb{R}^2$ with antidiscrete topology.

**Answer to E3.1. The first five topological spaces are homeomorphic:** we exhibit homeomorphisms between the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ and the other four spaces.

**Punctured plane** $\underset{f}{\overset{\sim}{\longrightarrow}}$ **annulus:** due to the rotational symmetry of both sets, it is convenient to define the continuous map $f$ **in polar coordinates.** See the diagram in the Figure.

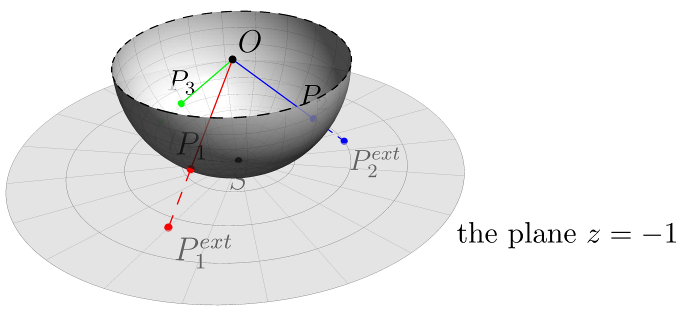
**Twice-punctured sphere** $\overset{\sim}{\longrightarrow}$ **punctured plane:** just use the stereographic projection, see Figure 3.2, which effects a homeomorphism between $\mathbb{R}^2$ and punctured sphere, and remains a homeomorphism if a point is removed from each space.




[AI]

**Punctured hemisphere $\xrightarrow{\sim}$ punctured plane:** modify the stereographic projection and project the hemisphere onto the plane from the origin $O$.

The diagram in the Figure shows how the homeomorphism is defined geometrically: if $P$ is a point on the hemisphere $\{(x,y,z) : x^2+y^2+z^2 = 1,\, z < 0\}$, extend the line $OP$ beyond $P$ and let $P^{ext}$ be the point of intersection of the extended line with the plane $\{z = -1\}$. The map $P \mapsto P^{ext}$ is the required homeomorphism, which remains such if the hemisphere and the plane are punctured by removing the point $S = (0,0,-1)$.




[AI]

[Link to online interactive 3D diagram]

**Punctured plane $\underset{g}{\xrightarrow{\sim}}$ cylinder:** we again use polar coordinates $(r,\theta)$ on the plane. We use **cylindrical** coordinates $(r,\theta,z)$ in $\mathbb{R}^3$, where the equation of the cylinder is $r = 1$.

Informally, the punctured plane consists of open half-lines extending radially from the origin. We would like to map each such half-line, which is isomorphic to $(0, +\infty)$, onto a straight line on the side of the cylinder (a *generatrix* of the cylinder).

A possible homeomorphism between $(0, +\infty)$ and the Euclidean line $\mathbb{R}$ is given by the mutually inverse functions $\ln$ and $\exp$:

$$(0, +\infty) \xrightarrow{\ln} \mathbb{R} \xrightarrow{\exp} (0, +\infty).$$

This results in the following homeomorphism $g\colon \mathbb{R}^2 \setminus \{(0,0)\} \to$ cylinder, polar to cylindrical coordinates:

$$g\colon (r, \theta) \mapsto (1, \theta, \ln(r)).$$

The inverse map is $g^{-1}\colon (1, \theta, z) \mapsto (e^z, \theta)$, and both $g$ and $g^{-1}$ are clearly continuous because $\ln$ and $\exp$ are continuous. Hence $g$ is a homeomorphism.

Since "homeomorphic" is an equivalence relation and in particular is transitive, we have done enough to show that the first five of the given spaces are pairwise homeomorphic. They are not homeomorphic to the remaining space, $\mathbb{R}^2$ with the antidiscrete topology. For example, they have the topological property that there exists an open set which is neither $\emptyset$ nor the whole space; $(\mathbb{R}^2$, antidiscrete$)$ does not have this property.

## References for the exercise sheet

**The homeomorphism $f(r, \theta)$ between the punctured plane and the annulus** was worked out by OpenAI ChatGPT by improving on two incorrect attempts. It is instructive to read the full conversation with the AI chatbot. The diagram is based on AI-generated code but enhanced visually by YB.

A full proof that $f$ is a homeomorphism can be read here. It is interesting to note that the AI-suggested formula, $r \mapsto \dfrac{r+2}{r+1}$, for the radial component of the map $f$ is a decreasing function. Thus, points of the punctured plane close to the (cut-out) origin are sent by $f$ to points on the annulus close to the outer boundary. Points of the punctured plane that are far away from the origin are sent by $f$ to points close to the inner boundary of the annulus.

**The homeomorphism between the punctured lower hemisphere and the punctured plane**: to produce the 3D diagram, the open hemisphere drawing by YB was used by OpenAI ChatGPT to add the tangent plane and the visual representation of sample points and their projections.

# Week 4

# Compactness

Version 2024/11/02 To accessible online version of this chapter

**Reminder:** an **open cover** of a topological space $X$ is a collection $\mathscr{C}$ of subsets of $X$ where, for each $U \in \mathscr{C}$, $U$ is an open subset of $X$, and $\bigcup \mathscr{C} = X$.

> **Definition: subcover of an open cover.**
>
> A **subcover** of an open cover $\mathscr{C}$ of $X$ is a subcollection of $\mathscr{C}$ which is still an open cover of $X$.

The following is one of the key notions of the course.

> **Definition: compact.**
>
> A topological space $X$ is **compact** if every open cover of $X$ has a finite subcover.

Compactness is a very powerful property, but it may require an effort to show directly that $X$ is compact, beyond simple examples. Let us start with a **non-example:**

> **Example: a non-compact topological space.**
>
> Show that the Euclidean line $\mathbb{R}$ is not compact.

**Solution:** consider the collection

$$\mathscr{C} = \{B_r(0) : r > 0\}$$

which consists of all open intervals $(-r, r)$ with $r$ positive. These intervals are open, and their the union contains all points of $\mathbb{R}$; that is, $\mathscr{C}$ is an open cover of $\mathbb{R}$.

Yet $\mathscr{C}$ has no finite subcover: any finite subcollection $\{B_{r_1}(0), \dots, B_{r_n}(0)\}$ of $\mathscr{C}$ has union equal to $B_R(0)$ where $R = \max(r_1, \dots, r_n)$, and this is not the whole of $\mathbb{R}$.

Thus, there is an open cover of $\mathbb{R}$ which has no finite subcover, so $\mathbb{R}$ is not compact.

At the moment, we can only give a very easy example of a compact space:

> **Example: a finite space is compact.**
>
> Let $X$ be a finite set. Show that any topology on $X$ is compact.

**Solution:** exercise.

> **Terminology.**
>
> We say **"a compact"** to refer to a compact topological space.
>
> We say **"$K$ is a compact set in $X$"** or **"a compact subset of $X$"** to mean that $K$ is a subset of a topological space $X$ such that $K$, viewed with the subspace topology, is compact.

We will often deal with compact sets contained inside some topological space, and the following technical lemma will simplify proofs.

> **Lemma 4.1: criterion of compactness for a subset.**
>
> Let $K$ be a subset of a topological space $X$. The following are equivalent:
>
> 1. $K$ is a compact subset of $X$.
> 2. Any collection $\mathscr{F}$ of open sets in $X$, which covers $K$ (that is, $K \subseteq \bigcup \mathscr{F}$), has a finite subcollection which still covers $K$.

*Proof (not given in class).* 1. $\Rightarrow$ 2.: suppose the subspace topology on $K$ is compact, and let $\mathscr{F}$ be a collection of **open subsets of** $X$ such that $K \subseteq \bigcup \mathscr{F}$. The collection $\mathscr{F}_K = \{U \cap K : U \in \mathscr{F}\}$ of subsets of $K$ is clearly an open cover of $K$. By assumption, $K$ is compact so this open cover must have a finite subcover, say $\{U_1 \cap K, \dots, U_n \cap K\}$. Then $\{U_1, \dots, U_n\}$ is a finite subcollection of $\mathscr{F}$ which still covers $K$.

2. $\Rightarrow$ 1.: to show that $K$ is compact, we let $\mathscr{C}$ be an open cover of $K$. By definition of subspace topology, $\mathscr{C}$ is of the form $\{U_\alpha \cap K : \alpha \in I\}$ where $U_\alpha$ are open in $X$. Clearly, for $\mathscr{C}$ to cover $K$, one must have $K \subseteq \bigcup_{\alpha \in I} U_\alpha$.

By condition 2., the collection $\{U_\alpha : \alpha \in I\}$ has a finite subcollection, say $U_{\alpha_1}, \dots, U_{\alpha_n}$, which still covers $K$. Hence $\mathscr{C}$ has finite subcover $U_{\alpha_1} \cap K, \dots, U_{\alpha_n} \cap K$, verifying the definition of "compact" for $K$. □

The next result shows that compactness is not only a topological property but can help solve Main Problem 2, mentioned earlier.

**Theorem 4.2: a continuous image of a compact is compact.**

If $X$ is a compact topological space and $f\colon X \to Y$ is continuous, then $f(X)$ is a compact set in $Y$.

*Proof.* We will use Criterion 4.1 of compactness for a subset to show that $f(X)$ is a compact set. Suppose a collection $\mathscr{G}$ of open sets in $Y$ covers $f(X)$: that is, $f(X) \subseteq \bigcup \mathscr{G}$.

Consider the collection $\mathscr{C} = \{f^{-1}(V) : V \in \mathscr{G}\}$ of subsets of $X$. We claim that $\mathscr{C}$ is an open cover of $X$. If $V$ is open in $Y$, "$f$ is continuous" means that $f^{-1}(V)$ is open in $X$, so all members of $\mathscr{C}$ are open in $X$. Also,

$$X = f^{-1}(f(X)) \subseteq f^{-1}\left(\bigcup \mathscr{G}\right) = \bigcup \{f^{-1}(V) : V \in \mathscr{G}\}$$

which shows that $\mathscr{C}$ covers $X$.

Since $X$ is compact, $\mathscr{C}$ has finite subcover, say $f^{-1}(V_1), \dots, f^{-1}(V_n)$. Then $V_1, \dots, V_n$ is a finite subcollection of $\mathscr{G}$ which covers $f(X)$. We have verified Criterion 4.1, hence $f(X)$ is a compact set in $Y$. □

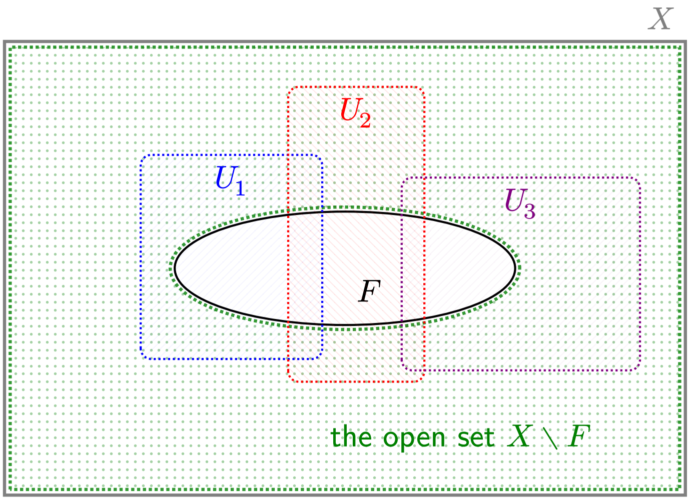



Figure 4.1: *if the sets $U_1, \dots, U_n, X \setminus F$ cover $X$, then the sets $U_1, \dots, U_n$ cover $F$*

**Corollary.**

Compactness is a topological property.

*Proof.* If $X$ is compact and $X \underset{f}{\overset{\sim}{\to}} Y$, then $f$ is continuous, so $f(X)$ must be compact by Theorem 4.2. Yet $f(X) = Y$ because $f$, being a homeomorphism, is surjective. □

If we found a compact space $X$, the next result allows us to construct new compact spaces.

**Proposition 4.3.**

A closed subset of a compact is compact.

*Proof.* Let $X$ be a compact topological space and let $F$ be a closed subset of $X$. We want to use Criterion 4.1, so we let $F$ be covered by a family $\mathscr{C}$ of open subsets of $X$. Then

$$\mathscr{C} \cup \{X \setminus F\}$$

is an open cover for the whole of $X$.

Since $X$ is compact, $\mathscr{C} \cup \{X \setminus F\}$ has a finite subcover of $X$. This finite subcover of $X$ can be $U_1, \ldots, U_n$ or $U_1, \ldots, U_n, X \setminus F$. In either case, the sets $U_1, \ldots, U_n$ form a finite subcollection of $\mathscr{C}$ which must cover $F$. (This last step is illustrated by Figure 4.1.) □

The above is as much as we can say about compact spaces without assuming additional topological properties besides compactness. We will now see that **compactness** works very well together with the **Hausdorff property:**

**Proposition 4.4.**

In a Hausdorff space, a compact set is closed.

*Proof.* Let $X$ be a Hausdorff topological space and let $K$ be a compact subset of $X$. Letting $z$ be any point of $X \setminus K$, it is enough to prove:

$$(\dagger) \qquad z \in X \setminus K \quad \Rightarrow \quad \exists \text{ open } V(z)\text{: } z \in V(z) \text{ and } V(z) \subseteq X \setminus K.$$

Indeed, if (†) holds then, in the same way as in the proof of Proposition 3.4 $X \setminus K = \bigcup_{z \in X \setminus K} V(z)$ is an open set, so $K$ is closed.

For each $x \in K$, the Hausdorff property gives us open neighbourhoods

$$U(x) \ni x, \quad V_x(z) \ni z : \quad U(x) \cap V_x(z) = \emptyset.$$

The open sets $\{U(x) : x \in K\}$ cover $K$, so by Criterion 4.1 there is a finite subcollection $U(x_1), \ldots, U(x_n)$ which still covers $K$. Put

$$V(z) = V_{x_1}(z) \cap \cdots \cap V_{x_n}(z).$$

(The construction of the open neighbourhood $V(z)$ is illustrated by Figure 4.2.)

As a finite intersection of open sets, $V(z)$ is open. Moreover, by construction $V(z) \cap U_{x_i}(z) \subseteq V_{x_i}(z) \cap U_{x_i}(z) = \emptyset$ and so

$$V(z) \cap (U(x_1) \cup \cdots \cup U(x_n)) = \bigcup_{i=1}^{n} V(z) \cap U(x_i) = \emptyset.$$

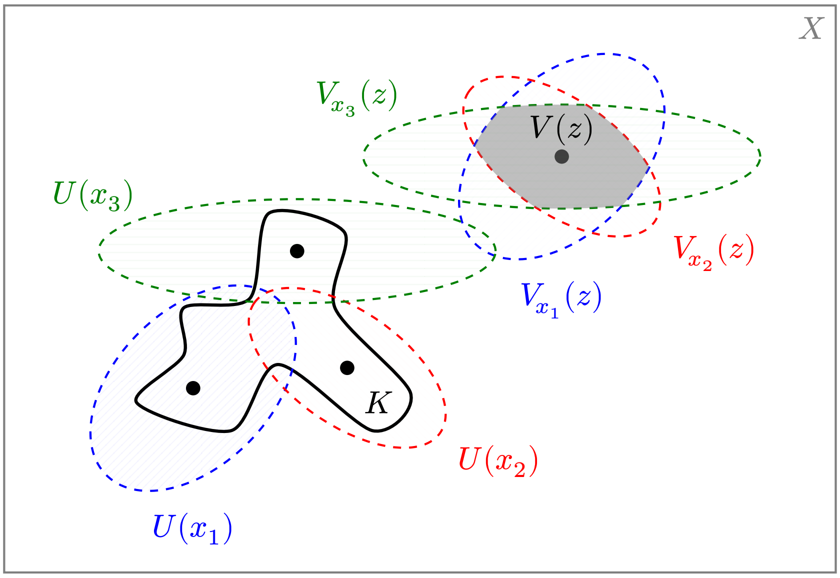



Figure 4.2: *the construction of the open neighbourhood $V(z)$ which does not intersect $K$*

Since $K$ is contained in the union $U(x_1) \cup \cdots \cup U(x_n)$, it follows that $V(z)$ does not intersect $K$. We have therefore proved (†) and the Proposition. □

We arrive at a result which generalises important results from real analysis known as inverse function theorems.

**Theorem 4.5: the Topological Inverse Function Theorem, $\mathscr{T}$IFT.**

If $K$ is a compact space, $Y$ is a Hausdorff space and $f\colon K \to Y$ is a continuous bijection, then $f$ is a homeomorphism.

*Proof.* $f$ is already assumed to be bijective and continuous, hence to show that $f$ is a homeomorphism, we need to prove that the inverse function $f^{-1}: Y \to K$ is continuous. We will use the closed set criterion of continuity, Proposition 2.5. Let $F \subset K$ be closed in $K$. The $f^{-1}$-preimage of $F$ is $(f^{-1})^{-1}(F) = f(F)$:

- a closed subset of a compact is compact (Proposition 4.3) so $F$ is compact,
- a continuous image of a compact is compact (Theorem 4.2), so $f(F)$ is compact,

- a compact subset of the Hausdorff space $Y$ is closed (Proposition 4.4), so $f(F)$ is closed in $Y$.

We have shown that the function $f^{-1}$ is such that the preimage of a closed set is closed. Hence, by the closed set criterion of continuity, $f^{-1}$ is continuous. □

## References for the week 4 notes

[Sutherland] gives detailed definitions of **cover, subcover** and **open cover** in [Sutherland, Definitions 13.3-13.5] and then defines a **compact subset** of $X$ straight away in [Sutherland, Definition 13.6], without defining a compact space first. In this way, [Sutherland] avoids the Criterion of Compactness for a subset 4.1 altogether — the Criterion becomes the definition of compactness!

Our Theorem 4.2, **a continuous image of a compact is compact,** is [Sutherland, Proposition 13.15].

The key idea behind **Figure 4.1** is by OpenAI ChatGPT (prompt: generate a diagram to illustrate the proof that a closed subset of a compact is compact). YB changed the shapes of sets to make the diagram less cluttered.

Proposition 4.3, **a closed subset of a compact is compact** is [Sutherland, Proposition 13.20]. Proposition 4.4, **a compact is closed in Hausdorff,** is [Sutherland, Proposition 13.12]. Theorem 4.5, **the topological inverse function theorem,** is [Sutherland, Proposition 13.26].

# Week 4

# Exercises (answers at end)

Version 2024/11/26. To accessible online version of these exercises

**Exercise 4.1** (basic test of openness)**.** Suppose that $\mathscr{B}$ is a base of a topology on $X$, and call the subsets of $X$ which are members of $\mathscr{B}$ **basic open sets.**

Let $A$ be a subset of $X$. Prove that the following are equivalent:

1. $A$ is open in $X$.
2. $A$ is a union of a collection of basic open sets.
3. For each point $x \in A$, there exists a basic open set $U$ such that $x \in U$ and $U \subseteq A$.

**Exercise 4.2** (the Euclidean topology has a countable base)**.** Consider the Euclidean space $\mathbb{R}^2$, and let $\mathscr{Q}$ be the (countable) collection of all open squares in $\mathbb{R}^2$ where the coordinates of all four vertices are rational numbers. Prove that $\mathscr{Q}$ is a base for the Euclidean topology.

Deduce that the collection of all open sets in the Euclidean space $\mathbb{R}^2$ has cardinality $\aleph$ (continuum), whereas the collection of all subsets of $\mathbb{R}^2$ has cardinality $2^{\aleph}$.

*Reminder about cardinal numbers:*

- $\aleph_0$ *(aleph-zero) denotes the countably infinite cardinality, e.g., the cardinality of* $\mathbb{N}$*;*

- $\aleph$ *(aleph) denotes the cardinality of continuum, e.g., the cardinality of* $\mathbb{R}$*,*
- *one has* $|\mathbb{R}| = \aleph = 2^{\aleph_0} = |P(\mathbb{N})| > \aleph_0$.

**Exercise 4.3** (subbase)**.** Let $(Y, \mathscr{T})$ be a topological space. A **subbase** of $\mathscr{T}$ is a collection $\mathscr{S}$ of open sets such that **finite intersections of sets from $\mathscr{S}$ form a base of $\mathscr{T}$.**

It is worth noting that, given any set $Y$ (without topology) and any collection $\mathscr{S}$ of subsets of $Y$, we can construct a topology $\mathscr{T}_{\mathscr{S}}$ on $X$ by using $\mathscr{S}$ as a subbase. That is, $\mathscr{T}_{\mathscr{S}}$ consists of arbitrary unions of finite intersections of members of $\mathscr{S}$. It is not difficult to show that this collection $\mathscr{T}_{\mathscr{S}}$ is a topology.

Prove that the collection of all **open rays** in the real line, i.e., sets of the form $(-\infty, a)$ and $(b, +\infty)$, is a subbase of the Euclidean topology.

**Exercise 4.4** (subbasic test of continuity)**.** Let $X$, $Y$ be topological spaces, $f\colon X \to Y$ be a function, and $\mathscr{S}$ be a subbase of topology on $Y$. Prove that the following are equivalent:

1. $f$ is continuous.
2. The preimage of every subbasic set in $Y$ is open in $X$ (meaning: $\forall V \in \mathscr{S}$, $f^{-1}(V)$ is open in $X$.)

**Exercise 4.5.** (a) Let $X$ be a topological space and let $f\colon X \to \mathbb{R}$ be a function. Prove: $f$ is continuous iff for all $a, b \in \mathbb{R}$, the sets $X_{f<a} = \{x \in X : f(x) < a\}$ and $X_{f>b} = \{x \in X : f(x) > b\}$ are open in $X$.

(b) Let $X$ be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous functions. Prove that the function $f + g\colon X \to \mathbb{R}$ is continuous. Hint: use (a).

# Week 4

# Exercises — solutions

Version 2024/11/26. To accessible online version of these exercises

**Exercise 4.1** (basic test of openness)**.** Suppose that $\mathscr{B}$ is a base of a topology on $X$, and call the subsets of $X$ which are members of $\mathscr{B}$ **basic open sets.**

Let $A$ be a subset of $X$. Prove that the following are equivalent:

1. $A$ is open in $X$.
2. $A$ is a union of a collection of basic open sets.
3. For each point $x \in A$, there exists a basic open set $U$ such that $x \in U$ and $U \subseteq A$.

**Answer to E4.1.** 1.$\iff$2. by definition of a base.

Proof that 2.$\Rightarrow$3.: assume that $A = \bigcup \mathscr{C}$ where $\mathscr{C}$ is a collection of basic open sets. Let $x \in A$. By definition of union of a collection of sets, $x$ is contained in at least one set in the collection $\mathscr{C}$; call this set $U$. The choice of $U$ ensures that

- $x \in U$;
- $U \in \mathscr{C}$, and $\mathscr{C}$ is a collection of basic open sets, so $U$ is a basic open set;
- $U \in \mathscr{C}$, and $\bigcup \mathscr{C} = A$, so $U \subseteq A$.

We have proved that 3. holds.

Proof that 3.⇒2.: assume that 3. holds. For each point $x \in A$, 3. allows us to choose a basic open set $U_x$ such that $x \in U_x \subseteq A$.

We claim that the union of the collection $\{U_x\}_{x\in A}$ of basic open sets is $A$. Indeed,

- for all $y \in A$, we have $y \in U_y$ by the choice of $U_y$, hence $y \in \bigcup_{x\in A} U_x$; this proves that $A \subseteq \bigcup_{x\in A} U_x$;
- for each $x \in A$, $U_x \subseteq A$, and so $\bigcup_{x\in A} U_x \subseteq A$.

Thus $\bigcup_{x\in A} U_x = A$, and so 2. holds.

**Exercise 4.2** (the Euclidean topology has a countable base)**.** Consider the Euclidean space $\mathbb{R}^2$, and let $\mathcal{Q}$ be the (countable) collection of all open squares in $\mathbb{R}^2$ where the coordinates of all four vertices are rational numbers. Prove that $\mathcal{Q}$ is a base for the Euclidean topology.

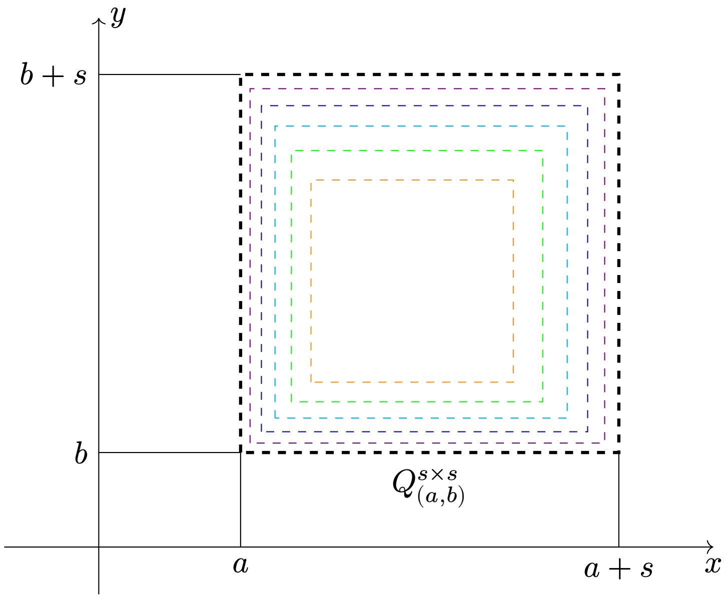
Deduce that the collection of all open sets in the Euclidean space $\mathbb{R}^2$ has cardinality $\aleph$ (continuum), whereas the collection of all subsets of $\mathbb{R}^2$ has cardinality $2^{\aleph}$.

**Answer to E4.2.** Denote by $Q^{s\times s}_{(a,b)}$ the square of size $s \times s$ whose bottom left corner is the point $(a,b)$ in $\mathbb{R}^2$.

1. First, we show that every square $Q^{s\times s}_{(a,b)}$ with $a, b, s$ real is a union of some collection $\{Q^{s_n\times s_n}_{(a_n,b_n)}\}_{n\geq 1}$ of squares with $a_n, b_n, s_n$ rational.

Indeed, let $(a_n)$ be a sequence of rational numbers such that $a_n \geq a$ and $\lim_{n\to\infty} a_n = a$. Also, let $(s_n)$ be a sequence of rational numbers such that $\lim s_n = s$ and $a_n + s_n \leq a + s$. (It is not difficult to show that such sequences of rational numbers exist.)

Then $Q^{s_n\times s_n}_{(a_n,b_n)} \subseteq Q^{s\times s}_{(a,b)}$ for all $n$, and $\bigcup_{n\geq 1}\{Q^{s_n\times s_n}_{(a_n,b_n)}\} = Q^{s\times s}_{(a,b)}$. See the Figure for illustration.




*Every square with sides parallel to the axes is a union of a collection of squares with rational coordinates*

2. Now we argue that every set which is open in the Euclidean plane $\mathbb{R}^2$ is a union of some open squares. Recall that an open square plays the role of $d_\infty$-open ball where the metric $d_\infty$ on $\mathbb{R}^2$ is defined by

$$d_\infty((x_1, x_2), (y_1, y_2)) = \max(|x_1 - y_1|, |x_2 - y_2|),$$

see the discussion after Proposition 2.3. Specifically, $Q^{s\times s}_{(a,b)}$ is the open ball $B^{d_\infty}_r((a + \frac{s}{2}, b + \frac{s}{2}))$ where $r = \frac{s}{2}$.

By definition of metric topology, open balls form a base of topology so it follows that every $d_\infty$-open set in $\mathbb{R}^2$ is a union of squares, and by 1., a union of rational squares.

It remains to recall that "$d_\infty$-open" is the same as "Euclidean open", because the metric $d_\infty$ is Lipschitz equivalent to the Euclidean metric $d_2$, see Proposition 2.3.

**Exercise 4.3** (subbase)**.** Let $(Y, \mathscr{T})$ be a topological space. A **subbase** of $\mathscr{T}$ is a collection $\mathscr{S}$ of open sets such that **finite intersections of sets from $\mathscr{S}$ form a base of $\mathscr{T}$.**

It is worth noting that, given any set $Y$ (without topology) and any collection $\mathscr{S}$ of subsets of $Y$, we can construct a topology $\mathscr{T}_{\mathscr{S}}$ on $X$ by using $\mathscr{S}$ as a subbase. That is, $\mathscr{T}_{\mathscr{S}}$ consists of arbitrary unions of finite intersections of members of $\mathscr{S}$. It is not difficult to show that this collection $\mathscr{T}_{\mathscr{S}}$ is a topology.

Prove that the collection of all **open rays** in the real line, i.e., sets of the form $(-\infty, a)$ and $(b, +\infty)$, is a subbase of the Euclidean topology.

**Answer to E4.3.** Let $\mathscr{S}$ be the collection of all open rays in $\mathbb{R}$. By taking intersections of just two sets from $\mathscr{S}$, we can generate all open bounded intervals in $\mathbb{R}$:

$$(b, a) = (-\infty, a) \cap (b, +\infty).$$

Since the open intervals $(b, a)$, where $a, b \in \mathbb{R}$, form a base of the Euclidean topology on $\mathbb{R}$, the topology $\mathscr{T}_{\mathscr{S}}$ generated by the subbase $\mathscr{S}$ contains the Euclidean topology.

On the other hand, every set in $\mathscr{S}$ is open in the Euclidean topology, hence so are unions of finite intersections of sets from $\mathscr{S}$. Therefore, the topology $\mathscr{T}_{\mathscr{S}}$ is contained in the Euclidean topology.

We conclude that $\mathscr{T}_{\mathscr{S}}$ is equal to the Euclidean topology, as claimed.

**Exercise 4.4** (subbasic test of continuity)**.** Let $X$, $Y$ be topological spaces, $f\colon X \to Y$ be a function, and $\mathscr{S}$ be a subbase of topology on $Y$. Prove that the following are equivalent:

1. $f$ is continuous.
2. The preimage of every subbasic set in $Y$ is open in $X$ (meaning: $\forall V \in \mathscr{S}$, $f^{-1}(V)$ is open in $X$.)

**Answer to E4.4.** 1. $\Rightarrow$ 2.: by definition of subbase, $\mathscr{S}$ is a subcollection of the topology on $Y$, i.e., every subbasic set in is open in $Y$. By definition of "continuous", the preimage of an open set is open, and so the preimages of subbasic sets must be open in $X$, proving 2.

2. $\Rightarrow$ 1.: a base $\mathscr{B}$ of topology on $Y$ consists of sets of the form $V_1 \cap \cdots \cap V_n$, where $n \geq 0$ and $V_1, \dots, V_n \in \mathscr{S}$. The preimage of intersection is the intersection of preimages, so we

have

$$f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n),$$

and, since $f^{-1}(V_i)$ is open in $X$ by 2., and a finite intersection of open sets is open, we conclude that $f^{-1}(V)$ is open in $X$ for all $V \in \mathscr{B}$.

Finally, every open set in $Y$ is a union of sets from $\mathscr{B}$, and the preimage of a union is the union of preimages. We conclude that $f^{-1}$(open set in $Y$) is open in $X$, hence, by definition of "continuous", $f$ is continuous, proving 1.

**Exercise 4.5.** (a) Let $X$ be a topological space and let $f\colon X \to \mathbb{R}$ be a function. Prove: $f$ is continuous iff for all $a, b \in \mathbb{R}$, the sets $X_{f<a} = \{x \in X : f(x) < a\}$ and $X_{f>b} = \{x \in X : f(x) > b\}$ are open in $X$.

(b) Let $X$ be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous functions. Prove that the function $f + g\colon X \to \mathbb{R}$ is continuous. Hint: use (a).

**Answer to E4.5.** (a) Note that $X_{f<a} = f^{-1}((-\infty, a))$ and $X_{f>b} = f^{-1}((b, +\infty))$. The sets $(-\infty, a)$ and $(b, +\infty)$ form a subbase of the Euclidean topology on $\mathbb{R}$ (see an earlier exercise). Hence by the subbasic test of continuity (see the previous exercise), $f$ is continuous iff all the sets $X_{f<a}$ and $X_{f>b}$ are open.

(b) We need to prove that the sets $X_{f+g<a}$, $X_{f+g>b}$ are open for all $a, b \in \mathbb{R}$. Note that

$$f(x) + g(x) < a \quad \iff \quad \exists t \in \mathbb{R} : \; f(x) < t, \; g(x) < a - t.$$

Indeed, $\Leftarrow$ is obvious, and to see $\Rightarrow$, take $t$ to be any real number in the interval $(f(x), a - g(x))$. The above rewrites in terms of sets as

$$X_{f+g<a} = \bigcup_{t \in \mathbb{R}} X_{f<t} \cap X_{g<a-t}.$$

Since $f, g$ are continuous, by (a) the sets $X_{f<t}$ and $X_{g<a-t}$ are open in $X$; the intersection of two open sets is open, and the union of any collection of open sets is open, which shows that $X_{f+g<a}$ is open.

It is shown in the same way that $X_{f+g>b}$ is an open subset of $X$, for all $b \in \mathbb{R}$. We now use (a) again to conclude that $f + g$ is a continuous function.

# Week 5

# Compactness in metric and Euclidean spaces. Connectedness

Version 2024/11/11 To accessible online version of this chapter

Metric spaces form a subclass of Hausdorff topological spaces. We can obtain further results about compact sets in this subclass. Recall the following from MATH21111 *Metric spaces*:

**Definition: bounded set**.

A subset $A$ of a metric space $(X, d)$ is **bounded** if $A \subseteq B_r(x)$ for some $x \in X$, $r > 0$.

We reproduce a result from MATH21111, but with a new proof which refers to the Hausdorff property:

**Proposition 5.1**.

In a metric space, a compact set is closed and bounded.

*Proof.* Let $K$ be a compact subset of a metric space $X$. The metric topology on $X$ is Hausdorff, Proposition 3.2, and compacts are closed in Hausdorff spaces, Proposition 4.4, so $K$ is closed in $X$.

To show that $K$ is bounded, fix any point $x$ of $X$ and consider the collection $\mathscr{C} = \{B_r(x)\}_{r\in\mathbb{R}_{>0}}$ of open balls. Clearly, $\bigcup\mathscr{C} = X$ and so $\mathscr{C}$ covers $K$. By Criterion of compactness 4.1, there exists a finite subcollection $\{B_{r_1}(x), \dots, B_{r_n}(x)\}$ of $\mathscr{C}$ which still covers $K$: that is,

$$K \subseteq B_{r_1}(x) \cup \dots \cup B_{r_n}(x) = B_R(x)$$

where $R = \max(r_1, \dots, r_n)$. We have shown that $K$ is a subset of an open ball, that is, $K$ is bounded. □

**Remark:** it is **not true** that every closed and bounded subset of a metric space is compact. An easy counterexample is given by $X$ endowed with a **discrete metric,** $d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$ The metric topology defined by $d$ on $X$ is the **discrete topology** which is **not compact is $X$ is infinite** (exercise).

A more conceptual example of a non-compact closed and bounded set in a metric space is the closed unit ball of an **infinite-dimensional Hilbert, or Banach, space.** This will be discussed in the second part of MATH31010, Functional Analysis.

The next result gives us a highly non-trivial example of a compact set in a Euclidean space.

**Theorem 5.2: the Heine-Borel Lemma.**

The closed bounded interval $[0, 1]$ is a compact subset of the Euclidean line $\mathbb{R}$.

*Proof.* There are several standard proofs of this result; we will present the proof by **bisection,** or **halving the interval.** Alternative proofs can be found in the literature.

Assume for contradiction that there exists a collection $\mathscr{C}$ of open subsets of $\mathbb{R}$ such that $[0, 1]$ is covered by $\mathscr{C}$, yet is not covered by any finite subcollection of $\mathscr{C}$. Then at least one of the two halves, the closed subintervals

$[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$

of $[0,1]$, has no finite subcover in $\mathscr{C}$: indeed, if $[0,\frac{1}{2}]$ were covered by finite $\mathscr{C}_1 \subseteq \mathscr{C}$, and $[\frac{1}{2},1]$, by finite $\mathscr{C}_2 \subseteq \mathscr{C}$, then the whole of $[0,1]$ would be covered by $\mathscr{C}_1 \cup \mathscr{C}_2$ which is a finite subcollection of $\mathscr{C}$. We therefore let

$$[a_1, b_1], \text{ where } 0 \le a_1 \le b_1 \le 1 \text{ and } b_1 - a_1 = \tfrac{1}{2},$$

denote one of the halves of $[0,1]$ which is not covered by any finite subcollection of $\mathscr{C}$. We can now apply the same argument to the closed bounded interval $[a_1, b_1]$ and obtain

$$[a_2, b_2], \text{ where } a_1 \le a_2 \le b_2 \le b_1 \text{ and } b_2 - a_2 = \tfrac{1}{2^2},$$

one of the halves of $[a_1, b_1]$ which is not covered by any finite subcollection of $\mathscr{C}$. Continuing this process, we will construct, for all $n \ge 1$, the interval

$$[a_n, b_n], \text{ where } a_{n-1} \le a_n \le b_n \le b_{n-1} \text{ and } b_n - a_n = \tfrac{1}{2^n},$$

which is not covered by any finite subcollection of $\mathscr{C}$.

Observe that the sequence $a_1 \le a_2 \le a_3 \le \ldots$ is increasing and bounded, as all its terms lie in $[0,1]$. By a result from Year 1 Foundations of Mathematics course, such a sequence converges to a limit $\ell$, and moreover $\ell \in [0,1]$, $a_n \le \ell$ for all $n$. Note also that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n + \frac{1}{2^n} = \ell$ and $\ell \le b_n$ for all $n$.

The point $\ell$ of $[0,1]$ must be covered by some set $U \in \mathscr{C}$. Since $U$ is open, $U \supseteq (\ell-\varepsilon, \ell+\varepsilon)$ for some $\varepsilon > 0$. Take $n$ such that $\frac{1}{2^n} < \varepsilon$. Then $\ell - \varepsilon < \ell - \frac{1}{2^n} \le a_n \le \ell$ and so $a_n \in (\ell-\varepsilon, \ell+\varepsilon)$. Similarly, $b_n \in (\ell-\varepsilon, \ell+\varepsilon)$, see Figure 5.1 for illustration.

Thus, the interval $[a_n, b_n]$ has a finite subcover in $\mathscr{C}$ — in fact, a cover by just one set, $U \in \mathscr{C}$ — contradicting the construction of $[a_n, b_n]$. This contradiction proves the Theorem. □

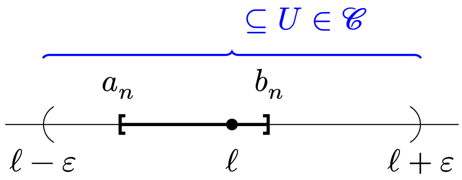



Figure 5.1: *the contradiction arrived at in the proof of the Heine-Borel Lemma*

**Remark:** it follows from the Heine-Borel Lemma that in every Euclidean space $\mathbb{R}^n$, a set is compact iff it is closed and bounded. This is because every such set is a subset of a cube of the form $[-M, M]^n \subseteq \mathbb{R}^n$. The closed bounded cube can be shown to be compact by adapting the subdivision argument in Theorem 5.2.

We will, however, not extend the subdivision argument to $n$ dimensions: in a short while, compactness of $[-M, M]^n$ will follow from the **(baby) Tychonoff theorem** which will say that a direct product of $n$ compact spaces is compact.

## Connectedness

So far, we have considered two topological properties: the Hausdorff property and compactness. The third topological property, and the final one that we will study in this course, is connectedness.

**Definition: connected.**

A topological space $X$ is **disconnected** if

$$\exists U, V \text{ open in } X\colon\ U \neq \emptyset,\ V \neq \emptyset,\ U \cap V = \emptyset,\ U \cup V = X.$$

That is, **a disconnected space is a disjoint union of two non-empty open sets.**

The space $X$ is **connected** if it is not disconnected.

Let us construct two **disconnected** spaces. We will obtain them as subspaces of the Euclidean line $\mathbb{R}$:

**Example.**

Show that the subspace $X = (0,1) \cup (2,3)$ of $\mathbb{R}$ is disconnected.

**Solution:** let $U$ be the open interval $(0,1)$ and $V$ be the open interval $(2,3)$. Then $U$ and $V$ are disjoint non-empty sets open in $X$ whose union is $X$. Hence, by definition, $X$ is disconnected.

**Example.**

Show that the subspace $X = \{0,1\}$ of $\mathbb{R}$ is disconnected.

**Solution.** Consider the following non-empty disjoint subsets of $X$: $U = \{0\}$ and $V = \{1\}$. We note that both sets are open in $X$. For example, $\{0\} = (-\infty, 1) \cap X$ where $(-\infty, 1)$ is an open subset of $\mathbb{R}$, and so $\{0\}$ is open in $X$ by definition of subspace topology. Similarly, $\{1\} = (0, +\infty) \cap X$.

We have written $X$ as a disjoint union of two non-empty open subsets, so by definition, $X$ is disconnected.

**Remark.** By showing $\{0\}$ and $\{1\}$ to be open in $\{0,1\}$, we have proved the following:

**Claim.**

The subspace $\{0,1\}$ of the Euclidean line $\mathbb{R}$ has discrete topology. □

One can say that the discrete space $\{0,1\}$ is the "canonical" example of a disconnected space. This point of view is supported by the following result.

**Proposition 5.3: conditions equivalent to connectedness.**

The following are equivalent for a topological space $X$:

(i) $X$ is connected.

(ii) For all continuous functions $f\colon X \to \mathbb{R}$, the image $f(X)$ is an interval.

(iii) Every continuous function $g\colon X \to \{0,1\}$ is constant.

Here $\mathbb{R}$ has Euclidean topology, and $\{0,1\}$ has discrete topology.

Before proving the Proposition, we formally define "interval".

**Definition: interval.**

An **interval** is a subset $I$ of $\mathbb{R}$ such that

$$\forall x, y \in I, \quad \forall t, \quad x < t < y \implies t \in I.$$

Thus, **together with any two points, an interval contains all intermediate points.**

It is not difficult to establish the following classification of intervals:

**Claim: classification of intervals in $\mathbb{R}$.**

$I \subseteq \mathbb{R}$ is an interval $\iff$ $I$ is a set of the form $(a, b)$, $[a, b)$, $(a, b]$, $[a, b]$, $(a, +\infty)$, $[a, +\infty)$, $(-\infty, b)$ or $(-\infty, b]$ for some real $a, b$, or $I = \mathbb{R}$.

**Remark:** the empty subset $\emptyset$ of $\mathbb{R}$ can be written as $(a, a)$ and is an interval. Any singleton $\{a\} \subseteq R$ can be written as $[a, a]$ and is an interval.

*Sketch of proof of the Claim (not given in class).* It is clear that every set $(a, b)$, $[a, b)$, ... , $\mathbb{R}$, listed in the claim, is an interval. Conversely, assume that $I$ is an interval in $\mathbb{R}$ and let $a = \inf I$, $b = \sup I$. If both $a$ and $b$ are real numbers and not $\pm\infty$ and so $I$ is non-empty and bounded, one uses the definition of the "least upper bound" $(\sup)$ and the "greatest lower bound" $(\inf)$ to show that $(a, b) \subseteq I \subseteq [a, b]$, which leaves four possibilities for $I$. The remaining cases when one or both of $a, b$ is infinite are handled similarly. Details are left to the student. □

*Proof of Proposition 5.3.* Recall for use in the proof that the preimage of intersection is the intersection of preimages; same for the union and the complement.

(i) ⇒ (ii): to prove the contrapositive, we must assume that there exists a continuous function $f\colon X \to \mathbb{R}$ such that $f(X)$ is not an interval. This means that there are $a, b \in f(X)$ and a real number $t$ such that $a < t < b$ and $t \notin f(X)$. Consider

$$U = f^{-1}((-\infty, t)), \quad V = f^{-1}((t, +\infty)).$$

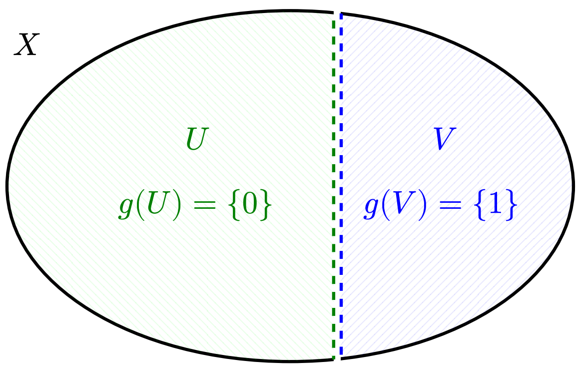



Figure 5.2: *defining* $g\colon X \to \{0,1\}$ *where* $X$ *is disconnected*

The set $U \subseteq X$ is open in $X$ because $U$ is the preimage of the set $(-\infty, t)$, open in $\mathbb{R}$, under a continuous function $f$. Furthermore, $U$ is not empty, as $f(U) \ni a$. Similarly, $V$ is open in $X$ and not empty; yet $U \cap V$ is

$$f^{-1}((-\infty,t)) \cap f^{-1}((t,+\infty)) = f^{-1}((-\infty,t) \cap (t,+\infty)) = f^{-1}(\emptyset),$$

so $U \cap V = \emptyset$ and $U, V$ are disjoint. Finally,

$$U \cup V = f^{-1}(\mathbb{R} \setminus \{t\}) = X \setminus f^{-1}(\{t\}),$$

yet $t \notin f(X)$ by assumption, so $f^{-1}(t) = \emptyset$ and $U \cup V = X$. The construction of $U, V$ shows that $X$ is disconnected by definition. We have shown not(ii) $\Rightarrow$ not(i).

(ii) $\Rightarrow$ (iii): let $g\colon X \to \{0,1\}$ be continuous. As in an earlier example, we can view the discrete set $\{0,1\}$ as a subspace of the Euclidean line $\mathbb{R}$. The inclusion map $\mathrm{in}\colon \{0,1\} \to \mathbb{R}$ is continuous by Proposition 2.7, and so we have a continuous map $X \xrightarrow{g} \{0,1\} \xrightarrow{\mathrm{in}} \mathbb{R}$. By (ii), the image of this map must be an interval in $\mathbb{R}$; yet this image is a subset of $\{0,1\}$, and such an interval can be at most one point. Thus, $g$ must be constant, proving (iii).

(iii) $\Rightarrow$ (i): to prove the contrapositive, assume that $X$ is disconnected, so that $X = U \cup V$ where $U$, $V$ are disjoint non-empty open sets. This allows us to define the following function (as illustrated in Figure 5.2):

$$g\colon X \to \{0,1\}, \quad g(x) = \begin{cases} 0, & x \in U, \\ 1, & x \in V. \end{cases}$$

We check that $g$ is continuous by calculating the preimage of every open subset of $\{0, 1\}$. There are exactly $4$ open subsets of the discrete space $\{0, 1\}$, and we have $g^{-1}(\emptyset) = \emptyset$, $g^{-1}(\{0\}) = U$, $g^{-1}(\{1\}) = V$ and $g^{-1}(\{0, 1\}) = X$. In each case, the preimage is an open subset of $X$, so $g$ is continuous by definition. We have constructed a non-constant continuous function $X \to \{0, 1\}$, proving not(i) $\Rightarrow$ not(iii). □

Here is an example where we use the Proposition 5.3 to establish that a space is connected.

> **Example: interval is connected.**
>
> Show that an interval $I \subseteq \mathbb{R}$ is connected.

**Solution:** we use condition (ii) from the Proposition. Let $f\colon I \to \mathbb{R}$ be a continuous function. To show that $f(I)$ is an interval, take two points $f(a)$ and $f(b)$ of $f(I)$, where $a, b \in I$. By the **Intermediate Value Theorem** from MATH11121 *Mathematical Foundations and Analysis,* all intermediate values between $f(a)$ and $f(b)$ belong to the image $f(I)$. Thus, by definition of "interval", $f(I)$ is an interval. Hence by Proposition 5.3, $I$ is connected, as claimed.

## References for the week 5 notes

The **bisection proof** of the **Heine-Borel Lemma** that we present in Theorem 5.2, is given in [Armstrong, Theorem (3.3)] and [Sutherland, Exercise 13.15]. An alternative "creeping-along" proof can be found in [Armstrong, Theorem (3.3)] and [Sutherland, Exercise 13.9].

Our **definition of "connected"** is identical to [Willard, Definition 26.1]. We thus deviate slightly from [Sutherland] which uses condition (iii) of our Proposition 5.3 as a definition [Sutherland, Definition 12.1], and turns our definition into a theorem, [Sutherland, Proposition 12.3]. The way [Armstrong] defines "connected", though equivalent to ours, requires the notion of closure of a set; but in our course closure comes after connectedness.

The definition of **interval** is elementary and is taken from [Smith, Definition 1.11]. The claim about classification of intervals is made in [Sutherland, Chapter 2: Notation and terminology].

# Week 5

# Exercises (answers at end)

Version 2024/10/30. To accessible online version of these exercises

**Exercise 5.1.** Let $A$ be a subspace of a topological space $X$. Prove: if $F \subseteq A$ and $F$ is closed in $X$, then $F$ is closed in $A$.

**Exercise 5.2** (unions and intersections of compact sets)**.** Let $X$ be a topological space.

1. Show that a union of two compact subsets of $X$ is compact.
2. Assuming that $X$ is Hausdorff, show that an intersection of two compact subsets of $X$ is compact. (Why do we need $X$ to be Hausdorff?)

**Exercise 5.3** (nested sequence of closed subsets in a compact)**.** Let $K$ be a compact topological space. Assume that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$, where $F_i$ is a non-empty closed subset of $K$ for each $i \geq 1$. Prove that all the sets $F_i$ have a common point.

**Exercise 5.4** (a Hausdorff compact topology is "optimal")**.** Let $(X, \mathscr{T})$ be a Hausdorff compact topological space. Use the Topological Inverse Function Theorem to show that

1. any topology on $X$, which is strictly weaker than $\mathscr{T}$, is not Hausdorff;
2. any topology on $X$, which is strictly stronger than $\mathscr{T}$, is not compact.

# Week 5

# Exercises — solutions

Version 2024/10/30. To accessible online version of these exercises

**Exercise 5.1.** Let $A$ be a subspace of a topological space $X$. Prove: if $F \subseteq A$ and $F$ is closed in $X$, then $F$ is closed in $A$.

**Answer to E5.1.** Assume that $F \subseteq A \subseteq X$ and that $F$ is closed as a subset of $X$, meaning that $X \setminus F$ is open in $X$. To show that $F$ is also closed as a subset of $A$, we write

$$A \setminus F = (X \setminus F) \cap A.$$

This means that by definition of subspace topology on $A$, the set $A \setminus F$ is open in $A$: this is a set of the form "(open in $X$) $\cap A$". Hence $F$ is closed in $A$.

**Exercise 5.2** (unions and intersections of compact sets)**.** Let $X$ be a topological space.

1. Show that a union of two compact subsets of $X$ is compact.
2. Assuming that $X$ is Hausdorff, show that an intersection of two compact subsets of $X$ is compact. (Why do we need $X$ to be Hausdorff?)

**Answer to E5.2.** Let $K$ and $L$ be two compact subsets of $X$.

1. We use Criterion of Compactness for Subsets 4.1 to show that $M = K \cup L$ is compact. Suppose that $M$ is covered by a collection $\mathscr{C}$ of sets open in $X$. Then $\mathscr{C}$ covers $K$ (because $K \subseteq M$). Since $K$ is compact, there exists a finite subcollection $U_1, \dots, U_m \in \mathscr{C}$ which still covers $K$.

In the same way, there exists a finite subcollection $V_1, \dots, V_n \in \mathscr{C}$ which covers $L$. Then the finite subcollection $U_1, \dots, U_m, V_1, \dots, V_n$ of $\mathscr{C}$ covers $K \cup L = M$. By constructing a finite subcollection of $\mathscr{C}$ which still covers $M$, we have verified the Criterion of Compactness for $M$; hence $M$ is compact.

2. In a Hausdorff space, a compact set is closed, Proposition 4.4, hence both $K$ and $L$ are closed in $X$.

Intersections of closed sets are closed, Proposition 2.4, so $K \cap L$ is closed in $X$.

By the previous exercise, $K \cap L$ is also closed in $K$. A closed subset of a compact is compact, Proposition 4.3, so $K \cap L$ is compact.

**Exercise 5.3** (nested sequence of closed subsets in a compact)**.** Let $K$ be a compact topological space. Assume that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$, where $F_i$ is a non-empty closed subset of $K$ for each $i \geq 1$. Prove that all the sets $F_i$ have a common point.

**Answer to E5.3.** Note that the question is about a collection of closed sets, whereas the definition of "compact" is in terms of open sets. The main idea is to **pass to the complement** and apply the De Morgan laws.

We are asked to prove that the intersection $\bigcap_{i=1}^{\infty} F_i$ is not empty. Equivalently, considering the complements $U_i = K \setminus F_i$ (which are **open**), we need to prove that **the union** $\bigcup_{i=1}^{\infty} U_i$ **is not the whole of $K$.**

Assume for contradiction that

$$\bigcup_{i=1}^{\infty} U_i = K.$$

Then the collection $U_1, U_2, \dots$ is an open cover of $K$. Since $K$ is compact, there is a finite subcover $U_{i_1}, \dots, U_{i_n}$ so that

$$U_{i_1} \cup \dots \cup U_{i_n} = K.$$

Now note that $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$, and therefore $U_{i_1} \cup \dots \cup U_{i_n} = U_j$ where $j = \max(i_1, \dots, i_n)$. We have

$$U_j = K \quad \Rightarrow \quad F_j = \emptyset.$$

Yet the sets $F_i$ were given to be non-empty. This contradiction shows that our **assumption** was false.

**Exercise 5.4** (a Hausdorff compact topology is "optimal")**.** Let $(X, \mathscr{T})$ be a Hausdorff compact topological space. Use the Topological Inverse Function Theorem to show that

1. any topology on $X$, which is strictly weaker than $\mathscr{T}$, is not Hausdorff;
2. any topology on $X$, which is strictly stronger than $\mathscr{T}$, is not compact.

**Answer to E5.4.** 1. A topology $\mathscr{T}_w$ on $X$ is weaker than $\mathscr{T}$ iff the identity function $\mathrm{id}_X \colon (X, \mathscr{T}) \to (X, \mathscr{T}_w)$ is continuous. (We note that the function $\mathrm{id}_X$ is always bijective.)

Assume that $(X, \mathscr{T})$ is compact and $(X, \mathscr{T}_w)$ is Hausdorff. Then by, $\mathscr{T}$IFT, the continuous bijection $\mathrm{id}_X \colon (X, \mathscr{T}) \to (X, \mathscr{T}_w)$ is a homeomorphism. That is, a set $U$ is $\mathscr{T}$-open iff its image $\mathrm{id}_X(U) = U$ is $\mathscr{T}_w$-open. But this means that $\mathscr{T}_w = \mathscr{T}$. Hence it is not possible for a Hausdorff $\mathscr{T}_w$ to be strictly weaker than $\mathscr{T}$.

Part 2. is done in a similar way and is left to the student.

## References for the exercise sheet

E5.1 is a variant of [Sutherland, Exercise 10.5]. E5.2 is [Sutherland, Exercises 13.3 and 13.10]. E5.3 is [Sutherland, Exercise 13.11].

# Week 7

# Connected components.

# Path-connectedness. Closure and interior

Version 2024/11/24 To accessible online version of this chapter

We continue to discuss connectedness.

**Terminology.**

We say **"$A$ is a connected set in $X$"** or **"a connected subset of $X$"** to mean that $A$ is a subset of a topological space $X$ such that $A$, viewed with the subspace topology, is connected.

**Theorem 7.1: a continuous image of a connected space is connected.**

If $X$ is a connected topological space and $f\colon X \to Y$ is continuous, then $f(X)$ is a connected set in $Y$.

*Proof.* Denote $Z = f(X)$. To prove that $Z$ is connected using Proposition 5.3(ii), we need to assume that $h\colon Z \to \mathbb{R}$ is a continuous function, and to show that $h(Z)$ is an interval in $\mathbb{R}$. Considering the composite function $h \circ f\colon X \xrightarrow{f} f(X) = Z \xrightarrow{h} \mathbb{R}$ which is continuous by Proposition 2.6, one has $h(Z) = (h \circ f)(X)$. By Proposition 5.3(ii), $(h \circ f)(X) \subseteq \mathbb{R}$ is an interval. We have shown that $h(Z)$ is an interval, as required. $\square$

**Remark** (*not made in the lecture*): strictly speaking, in the proof we replaced the function $f\colon X \to Y$ by the function $f\colon X \to Z = f(X)$, which is known as **restricting the codomain.** We have to explain why the restricted-codomain function $X \xrightarrow{f} Z$ is still continuous. But this is easy: if $V \subseteq Z$ is a set **open in** $Z$, then $V$ can be written as $Z \cap U$ where $U$ is open in $Y$. One has $f^{-1}(V) = f^{-1}(Z \cap U)$. The preimage of the intersection is the intersection of preimages, so this equals $f^{-1}(Z) \cap f^{-1}(U) = X \cap f^{-1}(U) = f^{-1}(U)$ which is open in $X$ as $X \xrightarrow{f} Y$ is given to be continuous. This shows that $X \xrightarrow{f} Z$ is continuous.

**Corollary**.

Connectedness is a topological property.

*Proof.* Replace the word "compact" with the word "connected" in the proof of the Corollary to Theorem 4.2. □

## Connected components

A topological space may be disconnected, yet it is always made of connected "pieces" called connected components. To define these, we recall the notion of equivalence relation.

**Notation**.

A **relation** on a set $X$ is any function $\sim\colon X \times X \to \{\text{True, False}\}$. We use infix notation for relations, writing "$\sim (x, y) = \text{True}$" as $x \sim y$ and "$\sim (x, y) = \text{False}$" as $x \nsim y$.

We have already verified the following definition for the relation "is homeomorphic to" on the class of all topological spaces. (Strictly speaking, this class is not a set, but we are going to ignore categorical subtleties here.) It is worth restating the definition more formally.

**Definition: equivalence relation, equivalence class.**

An **equivalence relation** relation on a set $X$ is a relation $\sim$ such that $\sim$ is

- **reflexive:** $\forall x \in X$, $x \sim x$;
- **symmetric:** $\forall x, y \in X$, $x \sim y \Rightarrow y \sim x$;
- **transitive:** $\forall x, y, z \in X$, $(x \sim y) \wedge (y \sim z) \Rightarrow x \sim z$.

Suppose the above holds. For each $x \in X$, the subset

$$[x] = \{y \in X : x \sim y\}$$

of $X$ is called the **equivalence class** of $x$.

We now introduce, on any topological space, an equivalence relation arising from connectedness.

**Proposition 7.2: equivalence relation $\sim$ given by connectedness.**

Let $X$ be a topological space. For $x, y \in X$, let $x \sim y$ mean "there exists a connected set $A \subseteq X$ such that $x, y \in A$". Then $\sim$ is an equivalence relation on $X$.

*Proof.* **We prove that $\sim$ is reflexive:** let $x \in X$. Put $A = \{x\}$. Then $A$ is a connected set: since $A$ consists of only one point, $A$ cannot be written as a union of two disjoint non-empty sets open in $A$. Since $x, x \in A$, we have $x \sim x$ by definition of $\sim$.

**We prove that $\sim$ is symmetric:** assume that $x, y \in X$ and $x \sim y$. Then there exists a connected set $A \subseteq X$ such that $x, y \in A$. The same can be written as $y, x \in A$, so $y \sim x$ by definition of $\sim$.

**We prove that $\sim$ is transitive:** assume that $x, y, z \in X$, $x \sim y$ and $y \sim z$. Then $x, y \in A$ and $y, z \in B$ where $A$ and $B$ are connected subsets of $X$. Note that $y \in A \cap B$ means that $A \cap B \neq \emptyset$, so by Lemma 7.3 below, the set $A \cup B$ is connected. Since $x, z \in A \cup B$, we have $x \sim z$ by definition of $\sim$. □

Here is the lemma used in the proof of transitivity of $\sim$.

**Lemma 7.3.**

If $A$, $B$ are connected subsets of $X$, $A \cap B \neq \emptyset$, then the union $A \cup B$ is connected.

More generally, if $\{A_\alpha : \alpha \in I\}$ is a collection of connected subsets of $X$ such that $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$, then the union $\bigcup_{\alpha\in I} A_\alpha$ is connected.

*Proof of the Lemma.* Pick a point $y \in A \cap B$. We will use Proposition 5.3(iii) to show that $A \cup B$ is connected.

Let $g\colon A \cup B \to \{0,1\}$ be any continuous function from $A \cup B$ to the discrete two-point space. The restriction $g|_A\colon A \to \{0,1\}$ is a continuous function on $A$: indeed, $g|_A = g \circ \mathrm{in}_A$, the inclusion map $\mathrm{in}_A$ is continuous by Proposition 2.7, and the composition of continuous maps is continuous by Proposition 2.6. Since $A$ is connected, by Proposition 5.3(iii) the function $g|_A$ is constant on $A$: all of its values on $A$ are equal to $g(y)$, that is, $g(A) = \{g(y)\}$.

In the same way, $g(B) = \{g(y)\}$. But then $g(A \cup B) = g(A) \cup g(B) = \{g(y)\}$. We have proved that $g$ is constant. This shows that $A \cup B$ is connected, by Proposition 5.3(iii).

The "more generally" part is proved similarly (*not in class*) and is left to the student. □

The equivalence classes defined by $\sim$ have a special name:

**Definition: connected components.**

Let $x \sim y$ be the relation "$\exists A \subseteq X$: $x, y \in A$, $A$ is connected" on a topological space $X$. The equivalence classes defined by $\sim$ are called **connected components** of $X$.

Recall that a **partition** of a set $X$ is a collection of subsets of $X$ which are non-empty, disjoint, and cover $X$. This is detailed in the following Claim, which is a well-known result from Mathematical Foundations.

**Claim: equivalence classes form a partition.**

If $\sim$ is an equivalence relation on a set $X$, the collection of equivalence classes $[x]$, where $x \in X$, forms a **partition** of the set $X$. That is,

- $[x]$ is non-empty for all $x$;
- either $[x] = [y]$ (equality of sets) or $[x] \cap [y] = \emptyset$, for all $x, y \in X$;
- $\bigcup_{x \in X} [x] = X$. □

**Corollary.**

Connected components of a topological space $X$ form a partition of $X$. That is, $X$ is a union of disjoint connected components.

The words "connected component" suggest that the set we are talking about is connected. This is indeed the case. *The following result was not proved in class.*

**Lemma 7.4.**

Each connected component of a topological space $X$ is a connected subset of $X$.

*Sketch of proof.* The connected component $[x]$ of a point $x \in X$ is the union of all connected sets $A$ in $X$ such that $x \in A$. The intersection of all such sets contains $x$, hence their union is connected by the second statement of Lemma 7.3. □

**Proposition 7.5: homeomorphism preserves connected components.**

If $h\colon X \xrightarrow{\sim} Y$ is a homeomorphism, $h$ maps connected components of $X$ to connected components of $Y$.

*Proof (not given in class).* Let $x \in X$. We denote the connected component of $x$ by $[x]$. Denote $y = h(x)$. Since $h$ is continuous, by Theorem 7.1 $h([x])$ is a connected subset of $Y$; it contains $y$, and so $h([x]) \subseteq [y]$.

Now, considering the continuous function $h^{-1}$, the same argument shows that $h^{-1}([y]) \subseteq [x]$, therefore $[y] \subseteq h([x])$. The two inclusions mean that $h([x]) = [y]$, as claimed. □

**Corollary.**

The number of connected components (or the cardinality of the set of connected components) is a topological property.

*Idea of proof (not given in class).* The Proposition implies that a homeomorphism $h\colon X \xrightarrow{\sim} Y$ defines a map $\{\text{connected components of } X\} \to \{\text{connected components of } Y\}$.

It is easy to see that this map must be a bijection, because $h$ is. Hence the set of connected components of $X$ must be equipotent with the set of connected components of any space homeomorphic to $X$. □

## Path-connectedness

We can see from Proposition 5.3 that connectedness of a topological space $X$ can be characterised in terms of functions **from** $X$ **to other spaces** such as $\mathbb{R}$ or $\{0, 1\}$. We will now consider a different topological property, expressed in terms of functions **to** $X$**.**

**Definition: path; points joined by a path.**

A **path** in a topological space $X$ is a a continuous function $\phi\colon [0, 1] \to X$. Points $x, y \in X$ are **joined by a path** if there exists a path $\phi$ with $\phi(0) = x$ and $\phi(1) = y$.

Here the closed interval $[0, 1]$ is considered with the Euclidean topology. A path should be thought of as a continuous curve in $X$ which starts at the point $x$ and ends at the point $y$, see Figure 7.1 for an illustration.

**Definition: a path-connected space.**

A space $X$ is **path-connected** if any two points of $X$ are joined by a path.

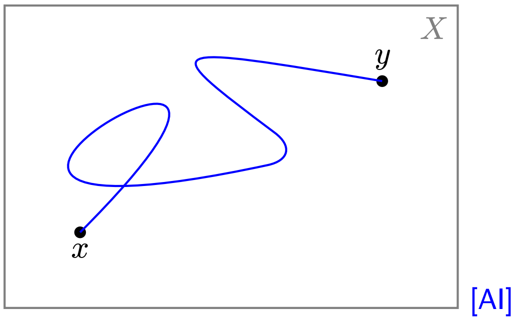



Figure 7.1: *"points $x$ and $y$ are joined by a path"*

**Claim.**

The continuous image of a path-connected space is path-connected. In particular, path-connectedness is a topological property.

*Proof (not given in class).* Suppose that $X$ is a path-connected space and $f\colon X \to Y$ is continuous. To show that $Z = f(X)$ is path-connected, we pick $a, b \in Z$. We have $a = f(x)$ and $b = f(y)$ for some $x, y \in X$. Now let $\phi\colon [0,1] \to X$ be a path with $\phi(0) = x$ and $\phi(1) = y$.

The function $f \circ \phi$, where $Z$ is taken as the codomain, is continuous, $(f \circ \phi)(0) = f(x) = a$ and $(f \circ \phi)(1) = f(y) = b$. Thus, $f \circ \phi$ is a path joining $a$ and $b$ in $Z$. $\square$

**Proposition 7.6: path-connected implies connected.**

If a topological space $X$ is path-connected, then $X$ is connected.

*Proof (not given in class).* Assume $X$ is path-connected, and fix a point $x \in X$. For any $y \in X$, let $\phi$ be a path joining $x$ and $y$. Then $x$ and $y$ lie in the set $\phi([0,1])$ which is a connected set, being a continuous image of the connected interval $[0,1]$. Hence $y$ lies in the connected component $[x]$ of $x$. Since $y$ was arbitrary, this shows that $X$ consists of only one connected component, and so $X$ is connected by Lemma 7.4. $\square$

**Example.**

Show that the Euclidean line $\mathbb{R}$ is not homeomorphic to the Euclidean plane $\mathbb{R}^2$.

**Solution** (*not given in class*)**:** assume for contradiction that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism. The set $\mathbb{R}^2 \setminus \{O\}$ is path-connected, see Figure 7.2: two points can be joined by a straight line segment or, if the segment contains $O$, by an arc; segments and arcs are paths. Since $f$ is injective, we have $f(\mathbb{R}^2 \setminus \{O\}) = \mathbb{R} \setminus \{f(O)\}$. Yet by Proposition 5.3(ii), a continuous image of a connected space must be an interval in $\mathbb{R}$, which $\mathbb{R} \setminus \{\text{point}\}$ is not. This contradiction shows that a homeomorphism $\mathbb{R}^2 \to \mathbb{R}$ does not exist.

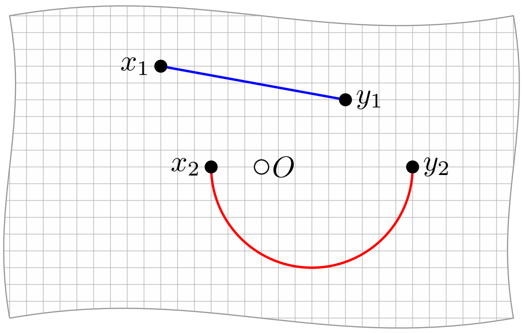



Figure 7.2: *the punctured plane* $\mathbb{R}^2 \setminus \{O\}$ *is path-connected*

## Closure and interior

We now extend two constructions, introduced in MATH21111 *Metric Spaces*, to general topological spaces.

**Definition: closure and interior of a set.**

Let $X$ be a topological space and $A$ be a subset of $X$. The **closure** of $A$ in $X$ is

$$\overline{A} = \bigcap\{F : A \subseteq F,\ F \text{ is closed in } X\}.$$

The **interior** of $A$ in $X$ is

$$A^\circ = \bigcup\{U : U \subseteq A,\, U \text{ is open in } X\}.$$

In the next result, the **smallest** set in some collection of sets is the set (if it exists) which is contained in all other sets of the collection. Likewise, the **largest** set in a collection is the set which contains all other sets of the collection.

**Claim 7.7.**

$\overline{A}$ is the smallest closed subset of $X$ which contains $A$.

$A^\circ$ is the largest open subset of $X$ contained in $A$.

*Proof.* Let us denote by $\mathscr{C}_A$ the collection $\{F : A \subseteq F,\, F \text{ is closed in } X\}$. Then $\overline{A}$ is defined as $\bigcap \mathscr{C}_A$. We need to prove statements 1,2,3 as follows:

1. **$\overline{A}$ is closed in $X$.** Indeed, $\mathscr{C}_A$ is a collection of closed sets, hence by Proposition 2.4(b), the intersection $\overline{A}$ of $\mathscr{C}_A$ is closed.
2. **$\overline{A}$ contains $A$.** Indeed, each set in $\mathscr{C}_A$ contains $A$, and so $\bigcap \mathscr{C}_A$ also contains $A$.
3. **$\overline{A} \subseteq G$ for all $G \in \mathscr{C}_A$.** Indeed, $\overline{A} = \bigcap \mathscr{C}_A = G \cap \bigcap\{F \in \mathscr{C}_A : F \neq G\}$. Since $\overline{A}$ is the intersection of $G$ with some set, we have $\overline{A} \subseteq G$, as claimed.

The claim about $A^\circ$ can be deduced from 1,2,3 above using the De Morgan laws 1.3: to do that, one shows that $A^\circ = X \setminus (\overline{X \setminus A})$. I leave this to the student. □

**Corollary.**

Let $A$ be a subset of a topological space $X$. Then

(1) $A$ is a closed set $\iff$ $A = \overline{A}$;

(2) $A$ is an open set $\iff$ $A = A^\circ$.

*Proof of Corollary.* (1) $\Rightarrow$: assume $A$ is closed in $X$. Then $A$ is a closed set which contains $A$. By Claim 7.7, $\overline{A}$ is the smallest such set, so $\overline{A} \subseteq A$. On the other hand, also by Claim 7.7, $\overline{A} \supseteq A$. The two inclusions show that $\overline{A} = A$.

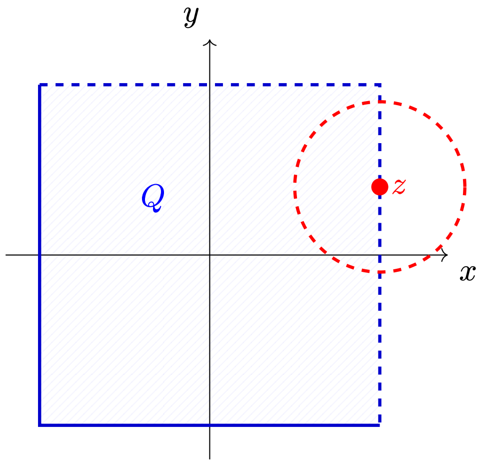



Figure 7.3: *$z$ is a limit point for the half-open square $Q = \{-1 \leq x < 1,\ -1 \leq y < 1\}$; $z \notin Q$ but $z \in \overline{Q}$*

$\Leftarrow$: assume $A = \overline{A}$. By Claim 7.7, $\overline{A}$ is closed. Hence $A$ is closed.

Part (2) is left to the student. □

## Closure as the set of "limit points"

We will now give another description of the closure of a set, based on the following:

> **Definition: limit point.**
>
> Let $A$ be a subset of a topological space $X$. A point $z \in X$ is a **limit point** for $A$ if $U \cap A \neq \emptyset$ for every open neighbourhood $U$ of $z$.
> In other words, **a point, whose every open neighbourhood meets $A$, is a limit point for $A$.**

It is obvious that if $z \in A$, then $z$ is a limit point for $A$. The converse is false in general, see Figure 7.3 for illustration.

**Proposition 7.8: closure equals the set of limit points.**

$\overline{A} = \{z \in X : z \text{ is a limit point for } A\}$.

*Proof.* We will prove: $y \notin \overline{A} \iff y$ is not a limit point for $A$.

$\Rightarrow$: assume $y \notin \overline{A}$. Then $y$ belongs to the set $U = X \setminus \overline{A}$. By Claim 7.7, $U$ is open (as $\overline{A}$ is closed) and $U$ does not meet $A$ (as $A \subseteq \overline{A}$). Hence, by definition of a limit point, $y$ is not a limit point for $A$, as claimed.

$\Leftarrow$: assume $y$ is not a limit point for $A$, so that there is open $U \ni y$ with $U \cap A = \emptyset$. Then $X \setminus U$ is closed, and $A \subseteq X \setminus U$. By Claim 7.7, $\overline{A} \subseteq X \setminus U$, and since $y \in U$, we conclude that $y \notin \overline{A}$. □

We note that our definition of a limit point is not in terms of sequences. We will now define limits of sequences, in order to see the connection with Real Analysis and Metric Spaces.

**Definition: convergence.**

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of a topological space $X$. We say that $x_n$ **converges to a point** $x \in X$, and write $x_n \to x$ as $n \to \infty$, if for any open neighbourhood $U$ of $x$ there exists $N \in \mathbb{N}$ such that the tail $x_{N+1}, x_{N+2}, \ldots$ of the sequence $(x_n)$ lies in $U$.

A sequence of points in a topological space may not converge to any point at all, converge to a single point, or converge to more than one point. This last option prevent us from saying "the limit of a sequence" because there might be more than one limit! This undesirable situation cannot occur in Hausdorff spaces:

**Proposition 7.9: in Hausdorff, limit is unique if it exists.**

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Hausdorff space $X$, such that $x_n \to x$ and $x_n \to y$ as $n \to \infty$. Then $x = y$.

*Proof (not given in class).* Assume for contradiction that $x \neq y$. Since $X$ is Hausdorff, $x \in U$ and $y \in V$ where $U, V$ are disjoint open sets.

Since $x_n \to x$, there exists $M \in \mathbb{N}$ such that $x_M, x_{M+1}, \cdots \in U$. Likewise, there exists $N \in \mathbb{N}$ such that $x_N, x_{N+1}, \cdots \in V$. But then $U$ and $V$ are not disjoint, because both sets contain $x_{\max(M,N)+1}$. This contradiction shows that the assumption $x \neq y$ was false. □

Let $A$ be a subset of a topological space $X$, and let $x \in X$. What is the relationship between the two statements,

(a) $x \in \overline{A}$;
(b) there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in A$ for all $n$, and $x_n \to x$ as $n \to \infty$.

In metric spaces, (a) and (b) are equivalent. In general topological spaces, (b) implies (a) but not the other way round. It turns out that the right condition for (a) and (b) to be equivalent is the following.

**Definition: a first-countable space.**

A topological space $X$ is **first countable** if every point $x \in X$ has a countable system $U_1(x), U_2(x), \ldots$ of open neighbourhoods, such that the collection $\{U_n(x) : n \geq 1,\ x \in X\}$ is a base of topology on $X$.

**All metric spaces are first countable:** just put $U_n(x) = B_{\frac{1}{n}}(x)$.

We omit the proof of the following fact, which the students may wish to attempt as an exercise or look up in the literature.

**Claim 7.10.**

If $X$ is a first-countable topological space and $A \subseteq X$, then $x \in \overline{A}$ iff there is a sequence $(x_n)_{n\in\mathbb{N}}$ contained in $A$ which converges to $x$. (In particular, this is true for all metrisable topologies.)

## The boundary of a set. Dense sets

We conclude the chapter with two definition which are important for normed, Hilbert and Banach spaces.

> **Definition: the boundary of a set.**
>
> Let $X$ be a topological space and $A \subseteq X$. The **boundary of** $A$ is the set $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

Combining this definition with Proposition 7.8, we arrive at the following equivalent description of the boundary of $A$:

> $\partial A$ is the set of points $z \in X$ such that every open neighbourhood of $z$ contains a point from $A$ and a point not from $A$.

In Euclidean spaces, the notion of the boundary is quite intuitive. For example, the boundary of the half-open square $Q = \{(x, y) : -1 \le x < 1,\ -1 \le y < 1\}$ in the plane is exactly the “border”, i.e., the union of the four sides, of the square: $\partial Q = \{(x, y) : \max(|x|, |y|) = 1\}$, see Figure 7.4 for illustration.

> **Definition: dense set.**
>
> Let $X$ be a topological space. A subset $A$ of $X$ is **dense** in $X$ if $\overline{A} = X$.

Of course, $X$ is always dense in $X$. Yet smaller (e.g., countable) dense sets, if they exist, are usually more interesting. The following is a standard example from *Metric Spaces*:

> **Example: $\mathbb{Q}$ is dense in $\mathbb{R}$.**
>
> Show that the set $\mathbb{Q}$ of rational numbers is dense in the Euclidean line $\mathbb{R}$.

**Solution** (not given in class)**:** let $z \in \mathbb{R}$ be arbitrary. We need to show that $z \in \overline{\mathbb{Q}}$, which by Proposition 7.8 means that every open neighbourhood $U$ of $z$ meets $\mathbb{Q}$. Indeed, by definition of “open” in Euclidean topology, $U$ contains an open interval $(z - \varepsilon, z + \varepsilon)$ for

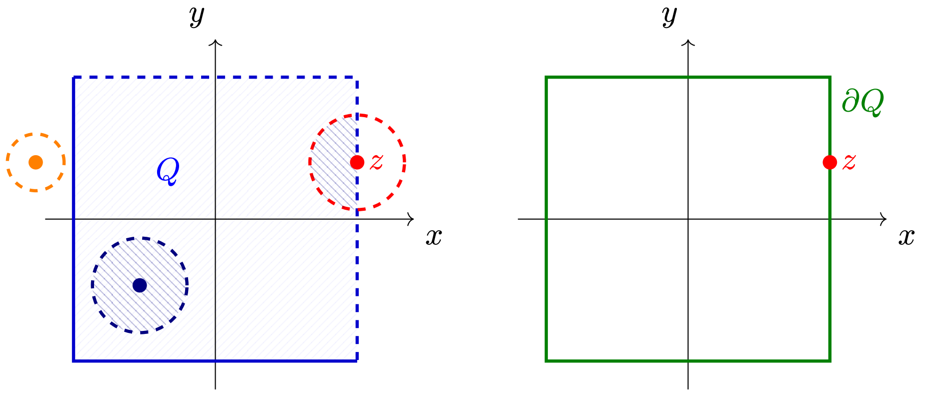



Figure 7.4: *the boundary of the half-open square* $Q = \{-1 \le x < 1, \ -1 \le y < 1\}$ *is the "border" of the square. Each open neighbourhood of a point on* $\partial Q$ *intersects both* $Q$ *and* $\mathbb{R}^2 \setminus Q$*. Non-boundary points have a neighbourhood fully in* $Q$ *or fully in* $\mathbb{R}^2 \setminus Q$

some $\varepsilon > 0$, and it is a known fact that every interval of positive length in $\mathbb{R}$ contains rational points. □

The concepts of "connected" and "dense" lead to a well-known counterexample in topology, which we will now consider.

## The rest of this chapter was not covered in class.

**Lemma 7.11.**

If a topological space $X$ has a connected dense subset, then $X$ is connected.

*Proof.* Let $A \subseteq X$ be such that $\overline{A} = X$. Assume that $X$ is disconnected: that is, $X = U \cup V$ where $U, V$ are disjoint non-empty sets open in $X$.

Take $x \in U$, so that $U$ is an open neighborhood of $x$. Since $x \in X = \overline{A}$, by Proposition 7.8 we must have $U \cap A \neq \emptyset$. Taking $y \in V$, we similarly argue that $V \cap A \neq \emptyset$. Then $A = (U \cap A) \cup (V \cap A)$ is a disjoint union of non-empty sets, open in $A$; hence $A$ is disconnected. The Lemma follows by contrapositive. □

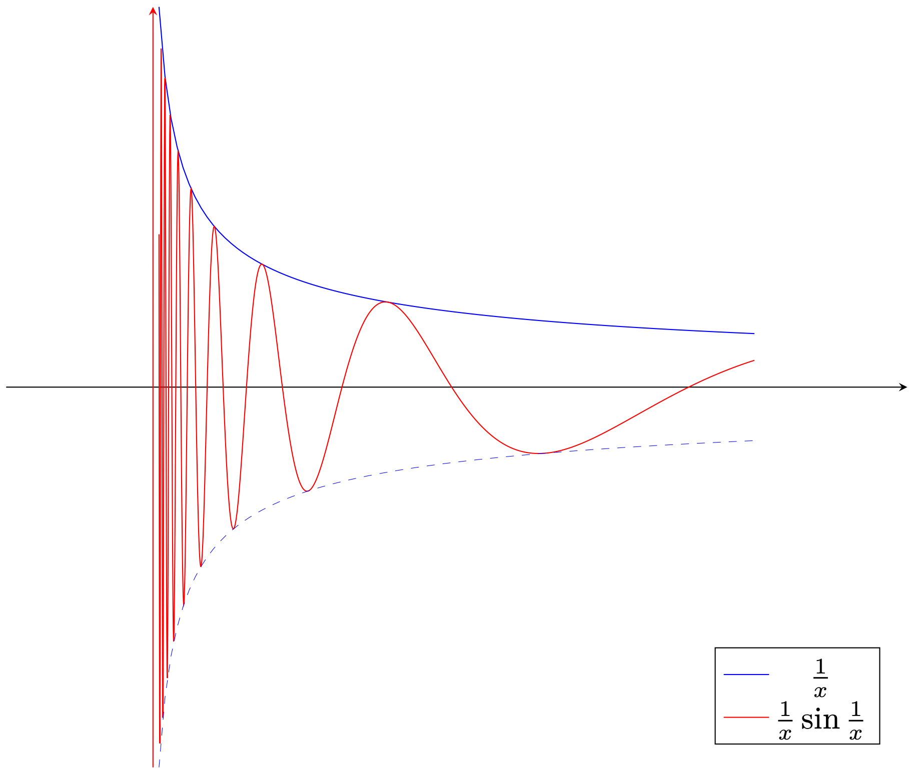



Figure 7.5: The (modified) topologist's sine curve

**Example: Topologist's sine curve.**

Let $X$ be the subet $\{(x, y) : x = 0$ or $x > 0, y = \frac{1}{x}\sin\frac{1}{x}\}$ of the Euclidean plane $\mathbb{R}^2$, see Figure 7.5. Show that $X$ is connected but not path-connected.

**Solution.** Let $X^+ = \{(x, y) : x > 0, y = \frac{1}{x}\sin\frac{1}{x}\}$ be the intersection of $X$ with the positive half-plane $\{x > 0\}$. Then $X^+$ is the image of $(0, +\infty)$ under the continuous function $x \mapsto (x, \frac{1}{x}\sin\frac{1}{x})$ from $(0, +\infty)$ to $\mathbb{R}^2$. Since the interval $(0, +\infty)$ is connected, and a continuous image of a connected space is connected (Theorem 7.1), $X^+$ is connected.

It is clear that every point of the vertical axis $\{x = 0\}$ is a limit point of $X^+$, thus $X = \overline{X^+}$, and by Lemma 7.11, $X$ is also connected.

Yet $X$ is not path-connected. Indeed, assume for contradiction that there is a path

$\phi\colon [0,1] \to X$ such that $\phi(0) = (0,0)$ and $\phi(1) = (1, \sin 1)$; here $(1, \sin 1)$ is a point of $X$. Denote by $p$ the projection $(x,y) \mapsto x$ which is continuous. Then $f = p \circ \phi$ is a continuous function $[0,1] \to [0, +\infty)$.

Since $[0,1]$ is connected, by Proposition 5.3(ii) $f([0,1])$ must be a real interval which contains $f(0) = 0$ and $f(1) = 1$. In particular, $f([0,1])$ contains $(0,1]$, which means that $\phi([0,1])$ contains a point of the form $(t,y)$ for all $t \in (0,1]$. Such a point of $X$ can only be $(t, \frac{1}{t}\sin\frac{1}{t})$. The $y$-coordinates of all such points are unbounded in $\mathbb{R}$, yet $\phi([0,1])$ must be compact by Theorem 4.2, hence bounded by Proposition 5.1. This contradiction shows that a path joining the points $(0,0)$ and $(1, \sin 1)$ inside $X$ does not exist. □

## References for the week 7 notes

Theorem 7.1, **a continuous image of a connected space is connected**, is [Sutherland, Proposition 12.11], and the Corollary (**connectedness is a topological property**) is [Sutherland, Corollary 12.12].

Topology textbooks, such as [Sutherland] and [Armstrong], assume knowledge of equivalence relations. This topic is covered in introductory mathematics literature: for example, [Smith] defines an **equivalence relation** (Definition 1.6), **partition** (Def.1.9), **equivalence class** $[x]$ (Def.1.10), and proves our Claim that **equivalence classes form a partition** [Smith, Proposition 1.4].

Figure 7.1 is a TikZ diagram generated with the help of OpenAI ChatGPT.

Definitions of **two points joined by a path** and a **path-connected space** are [Sutherland, Definitions 12.20 and 12.21]. Proposition 7.6, **path-connected implies connected**, is [Sutherland, Proposition 12.23], but we give a shorter proof. The example showing that $\mathbb{R}$ is not homeomorphic to the $\mathbb{R}^2$ is given in the book before [Sutherland, Exercise 12.1].

A **limit point** is called "a point of closure" in [Sutherland, Definition 9.6], and $\overline{A}$ is **defined** as the set of points of closure for $A$. Under this approach, our Proposition 7.8 is just the definition, yet our definition of $\overline{A}$ as the intersection of a family of closed sets becomes a result which needs proof; see [Sutherland, Proposition 9.10].

Proposition 7.9, **in Hausdorff, limit is unique if it exists**, is [Sutherland, Proposition 11.4].

Theorem 2.31 in the 2023/24 notes for MATH21111 *Metric Spaces* says: $y$ lies in $\overline{A}$ iff there exists a sequence $(y_n)_{n\geq 1}$ in $A$ such that $y_n \to y$ as $n \to \infty$.

First-countable spaces are defined in [Willard, Definition 10.3]. Claim 7.10 is [Willard, Thm 10.4].

The topologist's sine curve is a well-known example of a connected space which is not path-connected. It is given in [Counterexamples in Topology, 118], although we slightly modify it multiplying $\sin\frac{1}{x}$ by $\frac{1}{x}$ to arrive at an easier contradiction via unboundedness. A similar example under the same name is [Willard, Example 27.3a].

# Week 7

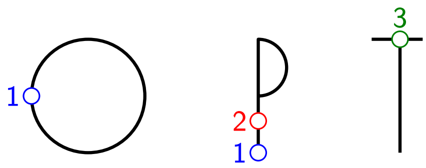
# Exercises (answers at end)

Version 2024/11/20. To accessible online version of these exercises

**Exercise 7.0. This is an unseen exercise in Applied Topology.** In the diagram below, each letter of the English alphabet is drawn as a union of straight line segments and arcs.

Some letters are homeomorphic: for example, C $\cong$ J, both are homeomorphic to a closed interval. **Consider such homeomorphisms to be geometrically obvious.**

Some letters are **not** homeomorphic: here is a topological property that can distinguish them. If $X$ is a topological space, call $p \in X$ **a point of connectivity** $k$ if $X \setminus \{p\}$ has exactly $k$ connected components. The following is easy to prove: *any homeomorphism* $X \xrightarrow{\sim} Y$ *maps a point of connectivity* $k$ *to a point of connectivity* $k$. Hence, for each $k$, **the number of points of connectivity $k$ is a topological property.** Example:




**O $\ncong$ P:** O has no points of connectivity 2 but P has them;
**T $\ncong$ O and T $\ncong$ P:** T has a point of connectivity 3 while O, P have no such points.

**CHALLENGE.** Sort the letters into homeomorphism classes. You should have 9 classes.

**Class 1:**

**Class 2:**

**Class 3:**

**Class 4:**

**Class 5:**

**Class 6:**

**Class 7:**

**Class 8:**

**Class 9:**

**Exercise 7.1.** Consider the topological space $\mathbb{Q}$ which is the set of all rational numbers, viewed as a subspace of the Euclidean real line $\mathbb{R}$.

1. Is $\mathbb{Q}$ Hausdorff? Is $\mathbb{Q}$ compact? Justify your answer.
2. Show that the topology on $\mathbb{Q}$ is **not** discrete.
3. A topological space $X$ is called **totally disconnected** if every non-empty connected subset of $X$ is a singleton. Show that $\mathbb{Q}$ is totally disconnected.

# Week 7

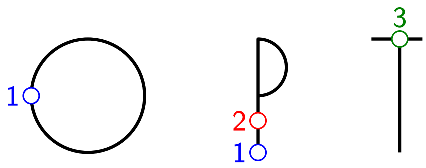
# Exercises — solutions

Version 2024/11/20. To accessible online version of these exercises

**Exercise 7.0. This is an unseen exercise in Applied Topology.** In the diagram below, each letter of the English alphabet is drawn as a union of straight line segments and arcs.


Some letters are homeomorphic: for example, C $\cong$ J, both are homeomorphic to a closed interval. **Consider such homeomorphisms to be geometrically obvious.**

Some letters are **not** homeomorphic: here is a topological property that can distinguish them. If $X$ is a topological space, call $p \in X$ **a point of connectivity** $k$ if $X \setminus \{p\}$ has exactly $k$ connected components. The following is easy to prove: *any homeomorphism* $X \xrightarrow{\sim} Y$ *maps a point of connectivity* $k$ *to a point of connectivity* $k$. Hence, for each $k$, **the number of points of connectivity** $k$ **is a topological property.** Example:




**O $\not\cong$ P:** O has no points of connectivity 2 but P has them;
**T $\not\cong$ O and T $\not\cong$ P:** T has a point of connectivity 3 while O, P have no such points.

**CHALLENGE.** Sort the letters into homeomorphism classes. You should have 9 classes.

**Class 1: have a point of connectivity 4:** K X

**Class 2: has two points of connectivity 3:** H

**Class 3: one point of connectivity 3, three points of connectivity 1:** E F T Y

**Class 4: one point of connectivity 3, infinitely many points of connectivity 1:** Q R

**Class 5: one point of connectivity 2:** B

**Class 6: the set of points of connectivity 2 is disconnected:** A

**Class 7: the set of points of connectivity 2 is connected:** P

**Class 8: intervals — two points of connectivity 1, all other points are of connectivity 2:** C G I J L M N S U V W Z

**Class 9: circles — all points are of connectivity 1:** D O

**Exercise 7.1.** Consider the topological space $\mathbb{Q}$ which is the set of all rational numbers, viewed as a subspace of the Euclidean real line $\mathbb{R}$.

1. Is $\mathbb{Q}$ Hausdorff? Is $\mathbb{Q}$ compact? Justify your answer.
2. Show that the topology on $\mathbb{Q}$ is **not** discrete.
3. A topological space $X$ is called **totally disconnected** if every non-empty connected subset of $X$ is a singleton. Show that $\mathbb{Q}$ is totally disconnected.

**Answer to E7.1.** 1. $\mathbb{Q}$ **is Hausdorff** because it is a subspace of a metric (hence Hausdorff) space $\mathbb{R}$. By Proposition 5.1, compacts in the metric space $\mathbb{R}$ must be closed and bounded. Since $\mathbb{Q}$ is not bounded, $\mathbb{Q}$ **is not compact.** (Another reason for non-compactness of $\mathbb{Q}$ is that $\mathbb{Q}$ is not closed in $\mathbb{R}$.)

2. Assume for contradiction that $\mathbb{Q}$ is discrete. Then $\{0\}$ must be an open subset of $\mathbb{Q}$, so by definition of subspace topology, $\{0\} = \mathbb{Q} \cap U$ where $U$ is open in $\mathbb{R}$. By definition of

an open set in a metric space, $U$ must contain the open ball $(-\varepsilon, \varepsilon)$ in $\mathbb{R}$ for some $\varepsilon > 0$. But any interval $(-\varepsilon, \varepsilon)$ of non-zero length contains infinitely many rational numbers, hence $\mathbb{Q} \cap U$ is infinite and not $\{0\}$. This contradiction shows that our assumption, "$\mathbb{Q}$ is discrete", was false.

3. Let $A \subseteq \mathbb{Q}$ be a non-empty connected set. The inclusion map $\mathrm{in}\colon \mathbb{Q} \to \mathbb{R}$ is continuous by Proposition 2.7, so $\mathrm{in}(A)$ is an interval in $\mathbb{R}$ by Proposition 5.3. Yet $\mathrm{in}(A) = A$, and non-empty intervals in $\mathbb{R}$ which consist entirely of rational points are singletons (every interval of non-zero length will contain irrationals). We have proved that $A$ is a singleton.

## References for the exercise sheet

E7.1 is an enhanced version of [Armstrong, Example 3 in Section 3.5].

# Week 8

# Definition of the product topology

Version 2024/11/26 To accessible online version of this chapter

Writing $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, we may define many metrics on $\mathbb{R}^2$, for example $d_1$, $d_2$ and $d_\infty$, see Figure 2.1. Yet, all these metrics define the same topology. This is not a coincidence: given topological spaces $X, Y$, we will now construct the standard topology $X \times Y$ called the product topology. Importantly, the construction extends to the Cartesian product of infinitely many topological spaces.

Key results of this chapter include the Tychonoff Theorem (only the baby version will be proved in class) and the Heine-Borel Theorem. We will consider one the topologists' favourite product space examples: the torus.

## The Cartesian product

We begin with a reminder about the Cartesian product of sets.

> **Definition: Cartesian product of two sets.**
>
> Let $X$, $Y$ be sets. The **Cartesian product** $X \times Y$ is the set of all pairs $(x, y)$ with $x \in X$, $y \in Y$.

The Cartesian product construction extends to arbitrary finite or infinite collections of sets:

- the Cartesian product of $n$ sets is a set of $n$-tuples,

$$X_1 \times \cdots \times X_n = \prod_{k=1}^{n} X_k = \{(x_1, \ldots, x_n) : x_k \in X_k \, \forall k = 1, \ldots, n\};$$

- for a sequence $X_1, X_2, \ldots$ of sets, the Cartesian product is a set of sequences,

$$\prod_{k=1}^{\infty} X_k = \{(x_k)_{k \geq 1} : x_k \in X_k \, \forall k \geq 1\};$$

- for a collection $\{X_\alpha : \alpha \in I\}$ of sets, the Cartesian product is a set of collections of elements indexed by $I$,

$$\prod_{\alpha \in I} X_\alpha = \{(x_\alpha)_{\alpha \in I} : x_\alpha \in X_\alpha \, \forall \alpha \in I\}.$$

We will initially focus on the Cartesian product of two sets.

Subsets of $X \times Y$ of a special form will be important to us:

**Definition: rectangle sets, cylinder sets.**

A **rectangle set** in $X \times Y$ is a set of of the form $A \times B$ where $A \subseteq X$ and $B \subseteq Y$.

A **cylinder set** in $X \times Y$ is a rectangle set of the form $A \times Y$ or $X \times B$.

Figure 8.1 illustrates these types of subsets of $X \times Y$. To produce such informal illustrations, one often visualises $X$ and $Y$ as intervals on the coordinate axes, and subsets $A$, $B$ as subintervals; this motivates the terminology.

Note that not all subsets of $X \times Y$ are cylinder or rectangle sets.

**Example: intersections of rectangle sets.**

Show that the intersection of any collection of rectangle sets is a rectangle set. Show that a union of rectangle sets may not be a rectangle set.

**Solution:** we calculate the intersection of two rectangle sets $A \times B$ and $A' \times B'$ where $A, A' \subseteq X$ and $B, B' \subseteq Y$. We have

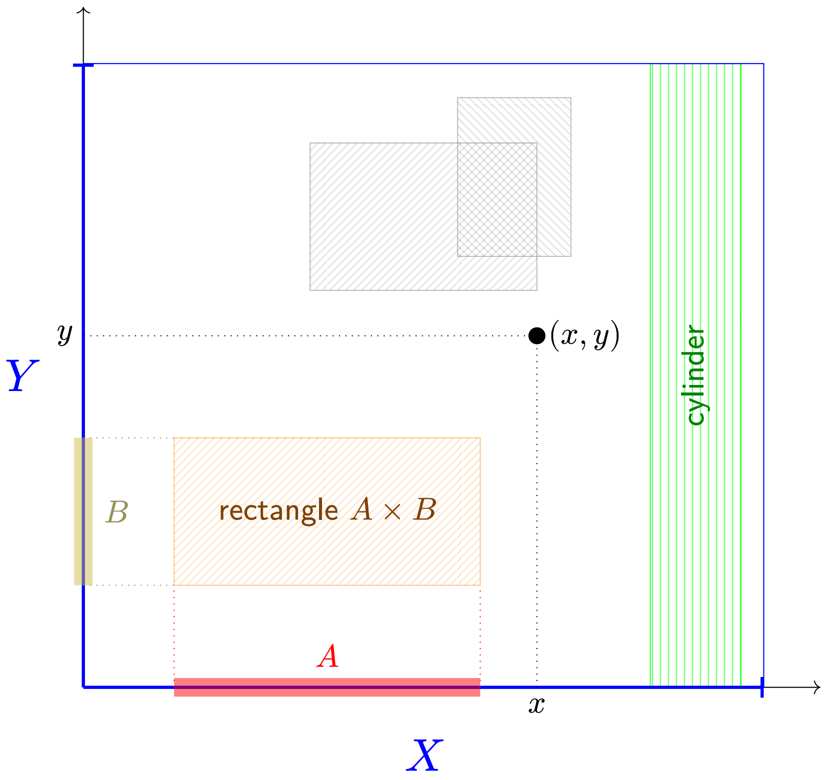



Figure 8.1: *rectangle and cylinder sets in $X \times Y$. The union of rectangles may not be a rectangle, but the intersection always is.*

$$(A \times B) \cap (A' \times B') = \{(x, y) : (x \in A \text{ and } y \in B) \text{ and } (x \in A' \text{ and } y \in B')\}$$
$$= \{(x, y) : x \in A \text{ and } x \in A' \text{ and } y \in B \text{ and } y \in B'\}$$
$$= (A \cap A') \times (B \cap B'), \text{ a rectangle set.}$$

In the same way one shows, for any collection $\{A_\alpha \times B_\alpha\}_{\alpha \in I}$ of rectangle sets, that

$$\bigcap_{\alpha \in I} (A_\alpha \times B_\alpha) = (\bigcap_{\alpha \in I} A_\alpha) \times (\bigcap_{\alpha \in I} B_\alpha),$$

i.e., the intersection is a rectangle set. Yet Figure 8.1 shows an example of two rectangle sets (with grey pattern) whose **union** is **not** a rectangle set.

# The product topology

From now on, we assume that $X$ and $Y$ are not just sets but **topological spaces.** Consider the collection

$$\mathscr{B} = \{U \times V : U \subseteq X \text{ is open in } X,\ V \subseteq Y \text{ is open in } Y\}$$

of subsets called **open rectangles** in $X \times Y$.

**Definition: product topology on $X \times Y$.**

The **product topology** on $X \times Y$ is the topology with base $\mathscr{B}$ of open rectangles. The set $X \times Y$ with this topology is the **product space** of $X$ and $Y$.

**Remark:** one needs to show that $\mathscr{B}$ is indeed a base of some topology. This means checking that the intersection of two sets from $\mathscr{B}$ can be written as a union of sets from $\mathscr{B}$. But here, an even stronger statement holds: $(U\times V)\cap(U'\times V') = (U\cap U')\times(V\cap V')$, that is, an intersection of two open rectangles is an open rectangle (no need to write it as a union of some collection of open rectangles).

We omit a full formal argument showing that $\mathscr{B}$ is a base of a topology; interested students can find it in the literature.

**Alert.**

Not all open sets in $X\times Y$ are open rectangles $U\times V$. Open sets are **arbitrary unions** of open rectangles.

The Euclidean plane is our first expected example of a product space.

**Example.**

Show that the metric Euclidean topology on $\mathbb{R}^2$ is the product topology on $\mathbb{R}\times\mathbb{R}$.

**Solution** (*not given in class*)**:** denote the product topology by $\mathscr{T}_{\mathbb{R}\times\mathbb{R}}$ and the metric Euclidean topology by $\mathscr{T}_{\text{metric}}$. Every open rectangle from the base $\mathscr{B}$ of $\mathscr{T}_{\mathbb{R}\times\mathbb{R}}$ is open in $\mathscr{T}_{\text{metric}}$, hence $\mathscr{T}_{\text{metric}}$ is stronger than $\mathscr{T}_{\mathbb{R}\times\mathbb{R}}$. On the other hand, we saw earlier that $\mathscr{T}_{\text{metric}}$ can be defined by the metric $d_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_2 - y_2|)$ and so has base of open squares $B_r((x, y)) = (x-r, x+r) \times (y-r, y+r)$ which is a subcollection of $\mathscr{B}$. Hence $\mathscr{T}_{\text{metric}}$ is weaker than $\mathscr{T}_{\mathbb{R}\times\mathbb{R}}$. We conclude that $\mathscr{T}_{\text{metric}} = \mathscr{T}_{\mathbb{R}\times\mathbb{R}}$. □

The product space $X \times Y$ comes equipped with two continuous maps.

**Proposition 8.1: projections are continuous.**

Given a product space $X \times Y$, the following **projection maps** are continuous:

$p_X \colon X \times Y \to X$, $(x, y) \mapsto x$, and

$p_Y \colon X \times Y \to Y$, $(x, y) \mapsto y$.

*Proof.* If $U \subseteq X$ is open, $p_X^{-1}(U) = U \times Y$. This is an open rectangle, hence an open set in $X \times Y$ by definition of the product topology. We have thus verified the definition of "$p_X$ is continuous". The proof for $p_Y$ is similar. □

## References for the week 8 notes

The definition of **product topology** on $X \times Y$ and a formal proof that the collection $\mathscr{B}$ of open rectangles in $X \times Y$ is a base of a topology are given in [Sutherland, Proposition 10.9]. Our Proposition 8.1, **the projections are continuous**, is [Sutherland, Proposition 10.10].

# Week 8

# Exercises (answers at end)

Version 2024/11/20. To accessible online version of these exercises

**Exercise 8.0. This is an unseen exercise on closure, boundary and dense sets.** Consider the sets $A = \{0,1\} \subset \mathbb{R}$ and $B = \mathbb{R} \setminus A = (-\infty, 0) \cup (0,1) \cup (1,+\infty)$ as a subsets of four different topological spaces, given in the table below. Complete the table.

| | The space $X$ | | | |
|---|---|---|---|---|
| | ($\mathbb{R}$, antidiscrete) | ($\mathbb{R}$, cofinite) | ($\mathbb{R}$, Euclidean) | ($\mathbb{R}$, discrete) |
| $\overline{A}$ (closure in $X$) | | | | |
| Is $A$ dense in $X$? (yes/no) | | | | |
| $\overline{B}$ | | | | |
| Is $B$ dense in $X$? (yes/no) | | | | |
| $\partial A$ | | | | |

*Hint.* You may wish to recall that $\overline{A}$ = the smallest closed set in $X$ which contains $A$ = $\{z \in X : \text{all open neighbourhoods of } z \text{ meet } A\}$ and that $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

**Exercise 8.1.** (a) Use the following two results,

- *a connected component of a topological space is a connected set* (Lemma 7.4),
- *if the space $X$ has a connected dense subset then $X$ is connected* (Lemma 7.11),

to show that each connected component of a topological space is a closed set.

(b) Deduce from (a) that if a topological space $X$ has finitely many connected components, then each connected component is both closed and open in $X$.

(c) Give an example of a topological space where connected components are closed but not open.

**Exercise 8.2.** (a) Suppose that $X$ is a topological space, points $x, y \in X$ are joined by a path in $X$, and points $y, z \in X$ are also joined by a path in $X$. Show that $x, z$ are joined by a path in $X$.

(b) Furthermore, show that "$x \sim y \iff x, y$ are joined by a path in $X$" is an equivalence relation on $X$.

Equivalence classes defined by the relation $\sim$ from (b) are called **path-connected components of $X$.** In general, a path-connected component does not need to be open or closed in $X$. Nevertheless:

(c) Show that if $X$ is an **open** subset of a **Euclidean space** $\mathbb{R}^n$, then each path-connected component of $X$ is open. Deduce that an open connected subset of $\mathbb{R}^n$ is path-connected.

# Week 8

# Exercises — solutions

Version 2024/11/20. To accessible online version of these exercises

**Exercise 8.0. This is an unseen exercise on closure, boundary and dense sets.** Consider the sets $A = \{0, 1\} \subset \mathbb{R}$ and $B = \mathbb{R} \setminus A = (-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ as a subsets of four different topological spaces, given in the table below. Complete the table.

| | The space $X$ | | | |
|---|---|---|---|---|
| | ($\mathbb{R}$, antidiscrete) | ($\mathbb{R}$, cofinite) | ($\mathbb{R}$, Euclidean) | ($\mathbb{R}$, discrete) |
| $\overline{A}$ (closure in $X$) | $\mathbb{R}$ | $A$ | $A$ | $A$ |
| Is $A$ dense in $X$? (yes/no) | yes | no | no | no |
| $\overline{B}$ | $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{R}$ | $B$ |
| Is $B$ dense in $X$? (yes/no) | yes | yes | yes | no |
| $\partial A$ | $\mathbb{R}$ | $A$ | $A$ | $\emptyset$ |

*Hint.* You may wish to recall that $\overline{A}$ = the smallest closed set in $X$ which contains $A$ = $\{z \in X : \text{all open neighbourhoods of } z \text{ meet } A\}$ and that $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

**Answer to E8.0.** Explanation of the above entries in the table: let $X$ be ($\mathbb{R}$, antidiscrete topology). The only closed sets in $X$ are $\emptyset$ and $\mathbb{R}$. Of these, only $\mathbb{R}$ contains $A$. Hence $\overline{A}$, which is the smalest closed set containing $A$, is $\mathbb{R}$. In the same way $\overline{B} = \mathbb{R}$. Since $\mathbb{R} \setminus A = B$, we have $\partial A = \overline{A} \cap \overline{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

Now let $X$ be ($\mathbb{R}$, cofinite topology). Note that the closed sets in the cofinite topology are finite sets and $\mathbb{R}$. Since $A$ is finite, $A$ is closed and $\overline{A} = A$. Since $B$ is infinite, the smallest closed set which contains $B$ is $\mathbb{R}$, hence $\overline{B} = \mathbb{R}$. We have $\partial A = A \cap \mathbb{R} = A$.

Next, let $X$ be ($\mathbb{R}$, Euclidean topology); this is a Hausdorff space, so singletons $\{0\}$ and $\{1\}$ are closed, and $A = \{0\} \cup \{1\}$ is closed as a finite union of closed sets. Hence $\overline{A} = A$. Every open neighbourhood of $0$ contains points from $B$, because an open set cannot consists just of points of the finite set $A$; hence $0 \in \overline{B}$. Similarly, $1 \in \overline{B}$. Hence $\mathbb{R} = \{0\} \cup \{1\} \cup B \subseteq \overline{B}$. We thus have $\overline{B} = \mathbb{R}$, and $\partial A = A \cap \mathbb{R} = A$.

Finally, if $X$ is ($\mathbb{R}$, discrete topology), then every subset of $X$ is closed and is equal to its own closure. So $\overline{A} = A$, $\overline{B} = B$ and $\partial A = A \cap B = \emptyset$.

In each case, a set is dense in $\mathbb{R}$ if its closure is $\mathbb{R}$.

**Exercise 8.1.** (a) Use the following two results,

- *a connected component of a topological space is a connected set* (Lemma 7.4),
- *if the space $X$ has a connected dense subset then $X$ is connected* (Lemma 7.11),

to show that each connected component of a topological space is a closed set.

(b) Deduce from (a) that if a topological space $X$ has finitely many connected components, then each connected component is both closed and open in $X$.

(c) Give an example of a topological space where connected components are closed but not open.

**Answer to E8.1.** (a) Let $x \in X$, and denote the connected component of $x$ by $C$. By Lemma 7.4, $C$ is connected. By definition of "dense", $C$ is dense in the subspace $\overline{C}$ of $X$. Therefore, by Lemma 7.11, $\overline{C}$ is connected. Since $\overline{C}$ contains $x$, this means that $\overline{C}$ must lies in the connected component of $x$, that is, $\overline{C} \subseteq C$. On the other hand, $C \subseteq \overline{C}$ for all sets $C$, see Claim 7.7. So $C = \overline{C}$, which is equivalent to $C$ being closed.

(b) Assume that $X$ is the union of finitely many connected components $C_1, \dots, C_n$. In part (a), we proved that $C_1, \dots, C_n$ are closed, and now we will show that $C_1$ is open. Recall that $C_1, \dots, C_n$ are equivalence classes, hence they are disjoint, and

$$C_1 = X \setminus (C_2 \cup \dots \cup C_n).$$

Finite unions of closed sets are closed, so $C_2 \cup \dots \cup C_n$ is closed, and its complement $C_1$ is open, as claimed.

(c) The set $\mathbb{Q}$ of rational numbers, viewed as the subspace of the Euclidean line $\mathbb{R}$, is totally disconnected, as shown in E7.1(3): every non-empty connected subset of $\mathbb{Q}$ is a singleton. Since connected components of $\mathbb{Q}$ are connected sets, it follows that **connected components of $\mathbb{Q}$ are singletons.** But a singleton (a one-point set) is **not open** in $\mathbb{Q}$: we showed exactly that in E7.1(2).

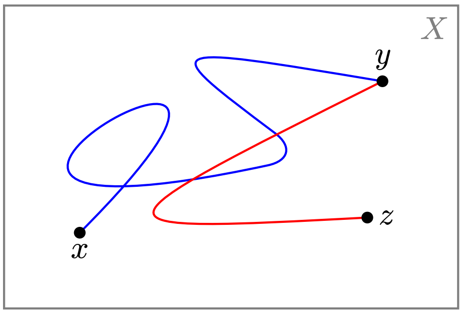
**Exercise 8.2.** (a) Suppose that $X$ is a topological space, points $x, y \in X$ are joined by a path in $X$, and points $y, z \in X$ are also joined by a path in $X$. Show that $x, z$ are joined by a path in $X$.

(b) Furthermore, show that "$x \sim y \iff x, y$ are joined by a path in $X$" is an equivalence relation on $X$.

Equivalence classes defined by the relation $\sim$ from (b) are called **path-connected components of $X$.** In general, a path-connected component does not need to be open or closed in $X$. Nevertheless:

(c) Show that if $X$ is an **open** subset of a **Euclidean space** $\mathbb{R}^n$, then each path-connected component of $X$ is open. Deduce that an open connected subset of $\mathbb{R}^n$ is path-connected.

**Answer to E8.2.** (a) Let $\phi$ be a path joining $x$ and $y$ in $X$. By definition, this means that $\phi\colon [0,1] \to X$ is a continuous function, $\phi(0) = x$ and $\phi(1) = y$. Similarly, a path joining $y$ and $z$ is a continuous function $\psi\colon [0,1] \to X$ with $\psi(0) = y$ and $\psi(1) = z$.




*A path joining $x$ and $z$ will be the blue path from $x$ to $y$ followed by the red path from $y$ to $z$; this is known as the concatenation of two paths*

We need to construct a continuous function $\chi\colon [0,1] \to X$ such that $\chi(0) = x$ and $\chi(1) = z$. Intuitively, we will construct a path which will first trace the path from $x$ to $y$, then trace the path from $y$ to $z$; this is called the **concatenation of paths** in the Figure, this will look like the continuous curve which is the union of the blue curve from $x$ to $y$ and the red curve from $y$ to $z$.

The functions $\hat{\phi}\colon [0,\frac{1}{2}] \to X$, $\hat{\phi}(t) = \phi(2t)$, and $\widehat{\psi}\colon [\frac{1}{2},1] \to X$, $\widehat{\psi}(t) = \psi(2t-1)$, are continuous as compositions of continuous functions. Define $\chi\colon [0,1] \to X$ by

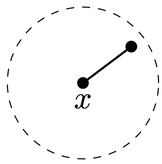
$$\chi(t) = \begin{cases} \hat{\phi}(t), & t \in [0,\frac{1}{2}], \\ \widehat{\psi}(t), & t \in [\frac{1}{2},1]. \end{cases}$$

Note that $\chi(\frac{1}{2})$ is well-defined as $\hat{\phi}(\frac{1}{2}) = \widehat{\psi}(\frac{1}{2}) = y$. Moreover, $\chi(0) = \phi(2 \times 0) = x$ and $\chi(1) = \psi(2 \times 1 - 1) = z$.

We prove that $\chi$ is continuous using the closed set criterion of continuity, Proposition 2.5. If $F \subseteq X$ is closed in $X$, $\chi^{-1}(F) = \hat{\phi}^{-1}(F) \cup \widehat{\psi}^{-1}(F)$. Since $\hat{\phi}$ is continuous, $\hat{\phi}^{-1}(F)$ is closed in $[0,\frac{1}{2}]$, hence compact. Similarly, $\widehat{\psi}^{-1}(F)$ is compact. A union of two compact sets is compact by E5.2, so $\chi^{-1}(X)$ is compact, hence closed in the Hausdorff space $[0,1]$. We have verified the closed set criterion of continuity for $\chi$. Thus, $x$ and $z$ are joined by the path $\chi$.

(b) Part (a) shows that the relation $\sim$ is transitive. We note that $\sim$ is reflexive: $x \sim x$ because the constant path, $\phi(t) = x$ for all $t \in [0,1]$, joins $x$ and $x$. Moreover, $\sim$ is symmetric: if $\phi$ is a path joining $x$ and $y$, then the path $\bar{\phi}\colon [0,1] \to X$, $\bar{\phi}(t) = \phi(t-1)$, joins $y$ and $x$. We conclude that $\sim$ is an equivalence relation.

(c) Suppose that $X \subseteq \mathbb{R}^n$ is an open set. If $x \in X$, take $r > 0$ such that $B_r(x) \subseteq X$. Note that in an open ball $B_r(x)$ in a Euclidean space, every point is joined to $x$ by a path — in fact, by a straight line segment. Hence $B_r(x)$ lies inside the path-connected component of $x$.




*In $\mathbb{R}^n$, any point of $B_r(x)$ can be joined by a straight-line path to $x$*

Thus, the path-connected component of each point of $X$ contains an open ball centred at that point. This is the definition of an open set in a metric space.

Now, if $X$ is a **connected** open set in $\mathbb{R}^n$, then $X$ cannot have more than one path-connected component: otherwise $X$ would be a union of disjoint **open** path-connected components, and a union of disjoint non-empty open sets is disconnected. Thus, $X$ consists of one path-connected component, and is path-connected.

## References for the exercise sheet

E8.1(a), **a connected component is closed,** is [Willard, Theorem 26.12]. The example in E8.1(c) is both [Armstrong, Example 3 in Section 3.5] and [Willard, Example 26.13a].

E8.2(a) (concatenation of paths) is based on [Sutherland, Lemma 12.2]. E8.2(c), **a connected open set in Euclidean space is path-connected,** is [Sutherland, Proposition 12.25] and [Willard, Corollary 27.6].

# Week 9

# Topological properties of product spaces

Version 2024/12/03 To accessible online version of this chapter

## The universal mapping property of the product space $X \times Y$

Any function $f\colon Z \to X \times Y$ must output a point of $X \times Y$ which is a pair, hence must be of the form $f(z) = (f_X(z), f_Y(z))$. Formally, the "components" of $f$ are

$$f_X : Z \xrightarrow{f} X \times Y \xrightarrow{p_X} X, \quad f_Y : Z \xrightarrow{f} X \times Y \xrightarrow{p_Y} Y$$

where $p_X, p_Y$ are the projections. The next result is called **the universal mapping property** because it describes **all** possible continuous functions with the codomain $X \times Y$.

**Theorem 9.1: the universal mapping property.**

Let $X, Y, Z$ be topological spaces. There is a 1-to-1 correspondence between

- continuous functions $f\colon Z \to X \times Y$, and
- pairs of continuous functions $(f_X\colon Z \to X, f_Y\colon Z \to Y)$.

To $f\colon Z \to X \times Y$ there corresponds the pair $(f_X = p_X \circ f, f_Y = p_Y \circ f)$. Vice versa, to a pair $(f_X, f_Y)$ there corresponds the function $f$ defined by $f(z) = (f_X(z), f_Y(z))$.

**Remark:** put simply, the Theorem says that $f\colon Z \to X \times Y$ is continuous **if and only if** the functions $f_X = p_X \circ f$, $f_Y = p_Y \circ f$ are continuous.

*Proof of the Theorem.* Let $f\colon Z \to X \times Y$ be a continuous function. Since $p_X, p_Y$ are continuous by Proposition 8.1, and a composition of continuous functions is continuous, the functions $f_X = p_X \circ f$, $f_Y = p_Y \circ f$ are continuous.

Vice versa, let pair $f_X\colon Z \to X$, $f_Y\colon Z \to Y$ be continuous functions. We need to check that the function $f = (f_X, f_Y)$ is continuous. By definition of the product topology, an arbitrary open subset of $X \times Y$ is a union $\bigcup_{\alpha\in I} U_\alpha \times V_\alpha$ of open rectangles, where, for each $\alpha \in I$, $U_\alpha$ is open in $X$ and $V_\alpha$ is open in $Y$. The preimage of a union is the union of preimages, so

$$f^{-1}(\bigcup_{\alpha\in I} U_\alpha \times V_\alpha) = \bigcup_{\alpha\in I} f^{-1}(U_\alpha \times V_\alpha).$$

Note that $f^{-1}(U_\alpha \times V_\alpha)$ consists of $z \in Z$ such that $f_X(z) \in U_\alpha$ and $f_Y(z) \in V_\alpha$. In other words, $f^{-1}(U_\alpha \times V_\alpha) = f_X^{-1}(U_\alpha) \cap f_Y^{-1}(V_\alpha)$ which is open in $Z$, because $f_X^{-1}(U_\alpha)$ and $f_Y^{-1}(V_\alpha)$ are preimages of open sets under the confinuous functions $f_X, f_Y$, and the intersection of two open sets is open.

Therefore, $\bigcup_{\alpha\in I} f^{-1}(U_\alpha \times V_\alpha)$ is open as a union of open sets. We have verified the definition of "continuous" for $f$.

It remains to note that, given $f\colon Z \to X \times Y$, taking $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$ then constructing $(f_X, f_Y)$ brings us back to the function $f$. Also, given the functions $f_X$, $f_Y$, if $f = (f_X, f_Y)$ then taking $p_X \circ f$ and $p_Y \circ f$ returns us to the functions $f_X$ and $f_Y$. This shows that we have two mutually inverse correspondences between functions $Z \to X \times Y$ and pairs of functions $Z \to X$, $Z \to Y$. Hence we have a bijective (1-to-1) correspondence between the two sets of functions, as claimed. $\square$

## Topological properties of the product space

We would like to understand how the topological properties of the product space $X \times Y$ are determined by the topological properties of the spaces $X$ and $Y$. The following result helps us by showing how homeomorphic copies of the space $X$ sit inside $X \times Y$:

**Lemma 9.2: embedding of $X$ in $X \times Y$.**

For each $y_0 \in Y$, the subspace $X \times \{y_0\}$ of $X \times Y$ is homeomorphic to $X$.

*Proof.* Consider the embedding map $i_{y_0}: X \to X \times Y$, $x \to (x, y_0)$. Note that $p_X \circ i_{y_0}$ is the identity map $x \mapsto x$ of $X$ which is continuous, and $p_Y \circ i_{y_0}$ is the constant map $\mathrm{const}_{y_0}: X \to Y$, which is continuous. Hence by the Universal Mapping Property, Theorem 9.1, $i_{y_0}$ is a continuous map. We can restrict the codomain and consider $i_{y_0}$ as a continuous map $X \to X \times \{y_0\}$.

The projection $p_X: X \times Y \to X$ is continuous by Proposition 8.1, so its restriction $p_X|_{X\times\{y_0\}}$ is continuous.

The composition $x \mapsto (x, y_0) \mapsto x$ shows that $p_X|_{X\times\{y_0\}} \circ i_{y_0} = \mathrm{id}_X$. Also, the composition $(x, y_0) \mapsto x \mapsto (x, y_0)$ shows that $i_{y_0} \circ p_X|_{X\times\{y_0\}} = \mathrm{id}_{X\times\{y_0\}}$. Therefore, $i_{y_0}$ and $p_X|_{X\times\{y_0\}}$ are two mutually inverse continuous maps. We have verified the definition of "homeomorphism" for $i_{y_0}: X \to X \times \{y_0\}$. □

**Remark:** in the same way, if $x_0 \in X$, the subspace $\{x_0\} \times Y$ of $X \times Y$ is homeomorphic to $Y$.

**Proposition 9.3: Hausdorfness and connectedness of the product.**

If $X, Y$ are non-empty topological spaces, then

(a) $X$ and $Y$ are both Hausdorff $\iff$ $X \times Y$ is Hausdorff;

(b) $X$ and $Y$ are both connected $\iff$ $X \times Y$ is connected.

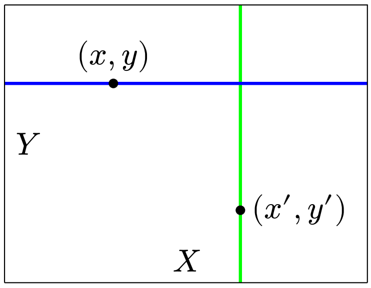



Figure 9.1: *if $X$, $Y$ are connected, any two points of $X \times Y$ are joined by a connected set*

*Proof (not given in class).* (a) $\Rightarrow$: assume $X$, $Y$ are Hausdorff, and let $(x, y) \neq (x', y')$ be two distinct points of $X \times Y$. Then either $x \neq x'$, or $y \neq y'$, or both.

If $x \neq x'$, take $U, U'$ to be open sets in $X$ such that $x \in U$, $x' \in U'$ and $U \cap U' = \emptyset$. Then the points $(x, y)$ and $(x', y')$ lie in disjoint open cylinder sets $U \times Y$ and $U' \times Y$. If $y \neq y'$, a similar argument leads to disjoint open cylinders $X \times V$ and $X \times V'$. We have thus verified the definition of "Hausdorff" for $X \times Y$.

$\Leftarrow$: assume $X \times Y$ is Hausdorff. Pick a point $y_0$ in the non-empty space $Y$. Subspaces of a Hausdorff space are Hausdorff (Proposition 3.3), so $X \times \{y_0\}$ is Hausdorff; it is homeomorphic to $X$ by Lemma 9.2, and Hausdorffness is a topological property (Proposition 3.1), so $X$ is Hausdorff. Similarly, $Y$ is Hausdorff.

(b) $\Rightarrow$: assume $X$, $Y$ are connected. Any two points $(x, y)$ and $(x', y')$ of $X \times Y$ lie in the set $(X \times \{y\}) \cup (\{x'\} \times Y)$, see Figure 9.1. Connectedness is a topological property, so $X \times \{y\}$, which is homeomorphic to $X$ by Lemma 9.2, is connected. Similarly, $\{x'\} \times Y$ is connected. The union of two connected sets, which have a common point, is connected by Lemma 7.3, so $(x, y)$ and $(x', y')$ lie in the same connected component of $X \times Y$. Since the two points were arbitrary, $X \times Y$ has only one connected component, i.e., is connected.

$\Leftarrow$: if $X \times Y$ is connected, then $X = p_X(X \times Y)$ and $Y = p_Y(X \times Y)$ are connected, because $p_X$, $p_Y$ are continuous (Proposition 8.1) and a continuous image of a connected space is connected (Theorem 7.1). $\square$

## The baby Tychonoff theorem about compactness of $X \times Y$

We will now deal with compactness of $X \times Y$ which is more intricate than Hausdorfness and connectedness. Since the product topology on $X \times Y$ is given by its base $\mathscr{B}$, we first prove a lemma which allows us to check compactness using only covers by sets from $\mathscr{B}$.

**Lemma 9.4: a "basic compact" is a compact.**

Let $\mathscr{B}$ be a base of topology on a space $X$. Suppose that every basic cover of $X$ (i.e., a cover by sets from $\mathscr{B}$) has a finite subcover. Then $X$ is compact.

*Proof.* By definition of a base, every open set $U \subseteq X$ can be written as a union of basic open sets (i.e., members of $\mathscr{B}$). Call these basic open sets "the children" of $U$.

Suppose that $\mathscr{C}$ is a cover of $X$ by open sets $U_\alpha$, $\alpha \in I$. Consider the collection $\mathscr{C}_1 = \{$all children of all sets from $\mathscr{C}\}$. Each set in $\mathscr{C}$ is the union of its children, so $\bigcup \mathscr{C}_1 = \bigcup \mathscr{C} = X$.

Thus, $\mathscr{C}_1$ is a basic open cover of $X$. Then by assumption, a finite cover $V_1, \dots, V_n$ can be chosen from $\mathscr{C}_1$: $V_1 \cup \dots \cup V_n = X$. The sets $V_1, \dots, V_n$ may not lie in $\mathscr{C}$, but their "parents" do. Consider the finite subcollection of $\mathscr{C}$ given by

a parent of $V_1$, a parent of $V_2$, ..., a parent of $V_n$.

Here "a parent" means an arbitrary choice of parent if $V_i$ has more than one parent. Since $V_i \subseteq$ (a parent of $V_i$), the union of the $n$ "parents" is also $X$. Given any open cover $\mathscr{C}$ of $X$, we constructed a finite subcover, thus verifying the definition of "compact" for $X$. $\square$

**Theorem 9.5: the baby Tychonoff theorem.**

If $X, Y$ are non-empty spaces, both $X$ and $Y$ are compact $\iff$ $X \times Y$ is compact.

*Proof.* $\Leftarrow$: $X \times Y$ is compact, so $X = p_X(X \times Y)$ is compact by Theorem 4.2 (continuous image of compact) as $p_X$ is continuous by Proposition 8.1. Similarly, $Y$ is compact.

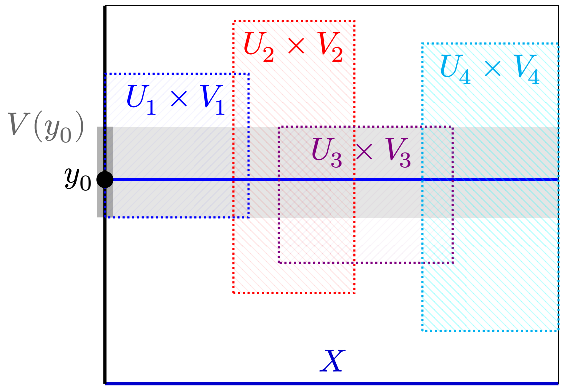



Figure 9.2: *if finitely many open rectangles cover $X \times \{y_0\}$, they cover an open cylinder $X \times V(y_0)$*

⇒: let $X$, $Y$ be compact. We assume that $\mathscr{C}$ is **a cover of $X \times Y$ by open rectangles** and show that $\mathscr{C}$ has a finite subcover. Since, by definition, open rectangles form a base of the topology on $X \times Y$, Lemma 9.4 will imply that $X \times Y$ is compact.

For each $y_0 \in Y$, $X \times \{y_0\}$ is homeomorphic to $X$ (Lemma 9.2), hence $X \times \{y_0\}$ **is a compact set,** which by compactness criterion 4.1 is covered by a finite subcollection of open rectangles $U_1 \times V_1, \dots, U_n \times V_n$ from $\mathscr{C}$. We may assume that each $U_i \times V_i$ intersects $X \times \{y_0\}$ (otherwise delete it from the finite cover), so $V_i \ni y_0$ for all $i = 1, \dots, n$.

Define the open neighbourhood $V(y_0)$ to be $V_1 \cap \cdots \cap V_n$. Then each open rectangle $U_i \times V_i$ contains $U_i \times V(y_0)$, and so the open rectangles $U_1 \times V_1, \dots, U_n \times V_n$, which form the finite cover of $X \times \{y_0\}$, **also cover the open cylinder** $X \times V(y_0)$, see Figure 9.2.

We have thus constructed an open neighbourhood $V(y_0)$ for **every point** $y_0$ **of** $Y$**.** We now use **compactness of** $Y$ to choose a finite subcover of $Y$ by these neighbourhoods: say, $V(y_1), \dots, V(y_m)$ such that $V(y_1) \cup \cdots \cup V(y_m) = Y$.

Then the union of the cylinders $X \times V(y_1), \dots, X \times V(y_m)$ is $X \times Y$. Also, by the above construction, each cylinder $X \times V(y_j)$ is covered by a finite subcollection of $\mathscr{C}$. The union of these $m$ finite subcollections is a **finite subcover, chosen from $\mathscr{C}$, for the whole of** $X \times Y$, as required. □

The following corollary is now easily obtained by induction. We understand the $n$-fold product $X_1 \times X_2 \times \cdots \times X_n$ as the iterated product $(((X_1 \times X_2) \times X_3) \times \dots) \times X_n$.

**Corollary: the product of finitely many compact spaces is compact.**

If $X_1, \dots, X_n$ are compact topological spaces, then the product space $X_1 \times \cdots \times X_n$ is compact. □

## The Heine-Borel Theorem

The baby Tychonoff theorem is now used to extend the Heine-Borel Lemma, Theorem 5.2, which says that $[0, 1]$ is compact, to a result which describes all compact sets in Euclidean spaces $\mathbb{R}^n$.

**Theorem 9.6: The Heine-Borel Theorem.**

In the Euclidean space $\mathbb{R}^n$, a set $K$ is compact iff $K$ is closed and bounded.

*Proof.* The "only if" part is immediate by Proposition 5.1: if $K$ is a compact set in any metric space, then $K$ is closed and bounded.

To prove the "if" part, assume that $K$ is a closed bounded subset of $\mathbb{R}^n$. Any bounded set is a subset of an $n$-dimensional cube $\{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| \leq M \ \forall i = 1, \dots, n\}$, for some $M > 0$. This cube of side length $2M$ is the product space $[-M, M] \times \cdots \times [-M, M] = [-M, M]^n$.

Note that the closed bounded interval $[-M, M] \subseteq \mathbb{R}$ is homeomorphic to $[0, 1]$ (a homeomorphism is afforded, for example, by a linear function mapping $[0, 1]$ onto $[-M, M]$). By the Heine-Borel Lemma, $[0, 1]$ is compact, and so $[-M, M]$ is also compact. By baby Tychonoff theorem 9.5 and its Corollary, $[-M, M]^n$ is compact.

Thus $K$ is a closed subset of the compact $[-M, M]^n$, and so by Proposition 4.3, $K$ is compact. □

## The torus and its embedding in $\mathbb{R}^3$

The theory that we have developed so far can be used to construct embeddings of abstractly defined topological spaces in a Euclidean space. By an embedding, we mean the following:

**Definition: embedding.**

Let $X$, $Y$ be topological spaces. An **embedding of $X$ in $Y$** is a map $f\colon X \to Y$ such that restricting the codomain gives a homeomorphism $f\colon X \xrightarrow{\sim} f(X)$.

In other words, an embedding means constructing inside $Y$ a subspace homeomorphic to $X$. This is important in many applications.

**Example: embedding of the torus in $\mathbb{R}^3$.**

Let $S^1$ denote the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in the Euclidean plane. Define the **torus** as the product space

$$\mathbb{T}^2 = S^1 \times S^1.$$

Show: $\mathbb{T}^2$ is compact. Construct an embedding of $\mathbb{T}^2$ in the Euclidean space $\mathbb{R}^3$.

**Solution:** $S^1$ is closed and bounded in $\mathbb{R}^2$, hence compact by Heine-Borel Theorem 9.6.

(There are alternative ways to show compactness of $S^1$; for example, $S^1$ is the image of the map $[0, 1] \to \mathbb{R}^2$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ which is continuous; $[0, 1]$ is compact, and a continuous image of a compact is compact.)

It follows that $\mathbb{T}^2 = S^1 \times S^1$ is compact by baby Tychonoff theorem 9.5.

It is not automatic that an embedding of $\mathbb{T}^2$ in $\mathbb{R}^3$ must exist: note that $S^1$ is a subspace of $\mathbb{R}^2$, so $\mathbb{T}^2$ is naturally a subspace of $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ and not $\mathbb{R}^3$. Yet we can construct an **injective** function $f\colon S^1 \times S^1 \to \mathbb{R}^3$, as follows.

A point of $\mathbb{T}^2$ is a pair $(P, Q)$ where $P$ is a point on the first circle, and $Q$ is a point on the second unit circle in $S^1 \times S^1$. It is convenient to represent the two points by their

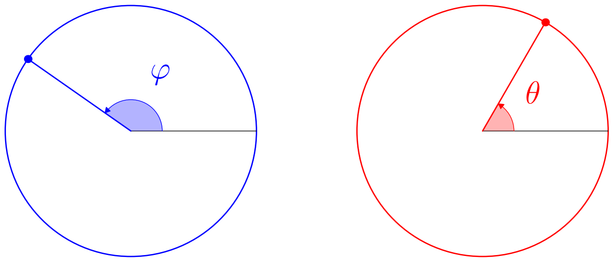



Figure 9.3: *the torus* $\mathbb{T}^2$ *is defined as the product of two circles, so a point of* $\mathbb{T}^2$ *is a pair of circle points and is represented by a pair of angles* $(\varphi, \theta)$

angle coordinates $(\varphi, \theta)$ with $\varphi, \theta \in [0, 2\pi)$, see Figure 9.3.

Fix two radii $R, r$ such that $R > r > 0$. Informally, we will think of the first circle in $S^1 \times S^1$ (the blue circle in Figure 9.3) as the circle of radius $R$ in the horizontal $xy$ plane in $\mathbb{R}^3$. The torus $\mathbb{T}^2$ is the disjoint union

$$\mathbb{T}^2 = \bigcup_{P \in S^1} \{P\} \times S^1,$$

and for each $P$ on the blue circle, we map the subset $\{P\} \times S^1$ of the torus to the red circle of radius $R$, centred at $P$ and orthogonal to the blue circle. Thus, $f\colon S^1 \times S^1 \to \mathbb{R}^3$ is given by

$$(\varphi, \theta) \mapsto \mathrm{Rotate}^{\varphi}_{z\text{-axis}}((R, 0, 0) + (\cos\theta, 0, \sin\theta))$$

as shown in Figure 9.4. An explicit formula for $f(\varphi, \theta)$ will be worked out in the tutorial.

**Explanation why $f$ is injective:** $f$ maps two points $(\varphi, \theta)$ and $(\varphi', \theta')$ on $\mathbb{T}^2$ such that $\varphi \neq \varphi'$, onto two disjoint red circles in Figure 9.4, to the $f$-images are distinct. The red circles are disjoint because we chose $r < R$. In the case $\varphi = \varphi'$, the two points are mapped by $f$ onto the same red circle but to different points of the circle, as long as $\theta \neq \theta'$.

**Explanation why $f$ is continuous:** the $x$-component $f_x$ of $f$ can be written as an algebraic expression in $\cos\varphi$, $\sin\varphi$, $\cos\theta$ and $\sin\theta$ (see the formula given in the tutorial). Note that $\cos\varphi$ and $\sin\varphi$ are the actual coordinates of the point on the first (blue) circle, hence they are continuous functions on $\mathbb{T}^2$ by Proposition 8.1. Same can be said of $\cos\theta$ and $\sin\theta$.

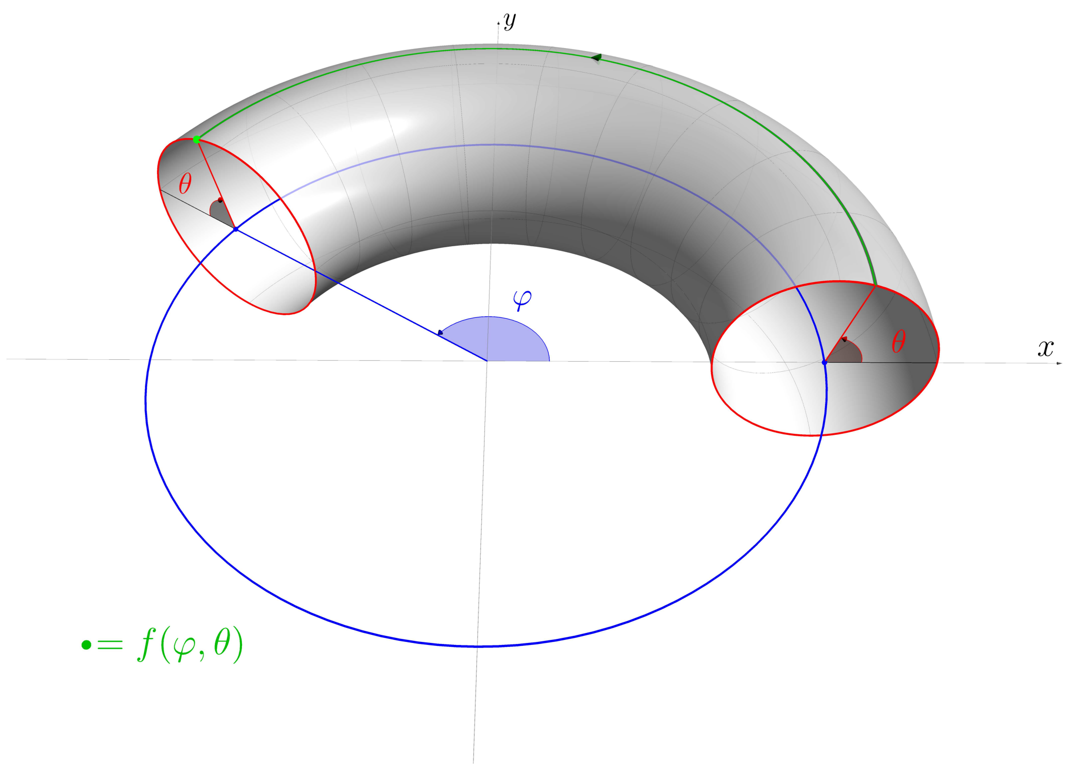



Figure 9.4: *the image of a point* $(\varphi, \theta) \in \mathbb{T}^2$ *in* $\mathbb{R}^3$ *is obtained by rotating, through* $\varphi$ *around the* $z$ *axis, the point* $\theta$ *on the red* $xz$ *circle of radius* $r$ *around* $(R, 0, 0)$. [Link to online interactive 3D diagram]

Sums and products of continuous functions are continuous (this is known to be true for metric spaces; for general topological spaces, see E4.5). Hence $f_x$ is a continuous function from $\mathbb{T}^2$ to $\mathbb{R}$.

In the same way one shows that $f_y$ and $f_z$ are continuous $\mathbb{R}$-valued functions on $\mathbb{T}^2$. Hence $f\colon \mathbb{T}^2 \to \mathbb{R}^3$ is continuous by the Universal Mapping Property, Theorem 9.1.

**Proof that $f$ is an embedding:** restricting the codomain of continuous injection $f$ gives the the continuous bijection $f\colon \mathbb{T}^2 \to f(\mathbb{T}^2)$. We **do not need continuity of the inverse map:** as $\mathbb{T}^2$ is compact and $f(\mathbb{T}^2)$ is metric hence Hausdorff, by the Topological Inverse Function Theorem 4.5 $f\colon \mathbb{T}^2 \xrightarrow{\sim} f(\mathbb{T}^2)$ is a homeomorphism. □

# The Tychonoff theorem (not done in class, not examinable)

Baby Tychonoff theorem 9.5 is a particular case of a result that we state here for completeness. The proof is beyond the scope of this course and can be found in the literature.

Let $(X_\alpha)_{\alpha \in I}$ be a collection of topological spaces. The topology on the Cartesian product $\prod_{\alpha \in I} X_\alpha$ is defined to have base $\mathscr{B}$ of sets of the form $\prod_{\alpha \in I} U_\alpha$ where (1) $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$; (2) $U_\alpha$ is open in $X_\alpha$ for all $\alpha \in I$.

The product topology on $X \times Y$ is a particular case: (1) can be omitted if $I$ is finite.

**Theorem 9.7: the Tychonoff theorem.**

Suppose the space $X_\alpha$ is not empty for all $\alpha \in I$. Then the product space $\prod_{\alpha \in I} X_\alpha$, defined above, is compact if, and only if, $X_\alpha$ is compact for all $\alpha \in I$. □

## References for the week 9 notes

Theorem 9.1, the Universal Mapping Property of $X \times Y$, is [Sutherland, Proposition 10.11] as well as [Armstrong, Theorem (3.13)].

Lemma 9.2 about a homeomorphic copy $X \times \{y_0\}$ of $X$ in $X \times Y$ is a strengthening of [Sutherland, Proposition 10.14] which only asserts that the map $i_{y_0} : x \mapsto (x, y_0)$ is continuous.

Proposition 9.3: (a) Hausdorffness of $X \times Y$ is [Sutherland, Proposition 11.17b], [Armstrong, Theorem (3.14)]; (b) connectedness is [Sutherland, Theorem 12.18], [Armstrong, Theorem (3.26)].

The baby Tychonoff theorem 9.5 is [Sutherland, Theorem 13.21] and [Armstrong, Theorem (3.15)]. Our proof follows [Armstrong]. In particular, our Lemma 9.4 is [Armstrong, Lemma (3.16)].

The Heine-Borel theorem 9.6 is [Sutherland, Theorem 13.22] and [Armstrong, Theorem (3.1)].

The general Tychonoff theorem 9.7 is not usually proved in introductory-level Topology textbooks. A proof can be found in [Willard, Theorem 17.8].

# Week 9

# Exercises — solutions

Version 2024/12/03. To accessible online version of these exercises

**Exercise 9.0. An unseen exercise on gluing. Definition:** let $X$ be a top. space, $Y$ a set (initially without a topology), $q\colon X \to Y$ a **surjective** map. The topology on $Y$ given by

$$V \subseteq Y \text{ is open in } Y \overset{\text{def}}{\iff} q^{-1}(V) \text{ is open in } X$$

is **the quotient topology** on $Y$ induced by $q$. In this situation, $q$ is **the quotient map.**

**In particular,** if $\sim$ is an equivalence relation on $X$, take $Y := X/{\sim}$ to be the set of $\sim$-equivalence classes, and define $q\colon X \to Y$ by $q(x) = [x]$. The quotient topology on $X/{\sim}$ is called **the identification topology** with respect to $\sim$. (Idea: whenever $x' \sim x''$, we identify points $x'$ and $x''$ and treat them as one point.)

**Gluing topology** is identification topology where $\sim$ is such that

- some equivalence classes consist of $2$ points (two points **glued together**);
- the rest of equivalence classes are singletons.

**Theorem (universal mapping property for a quotient space).** Let $Y$ be a quotient space via the quotient map $X \xrightarrow{q} Y$. Given any topological space $Z$, there is a 1-to-1 correspondence between

- continuous maps $f\colon Y \to Z$;
- continuous $F\colon X \to Z$ such that $F(x') = F(x'')$ in $Z$ whenever $q(x') = q(x'')$ in $Y$.

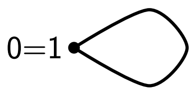
The correspondence is such that $q \circ F = f$.

**CHALLENGE:** construct embeddings, or at least "immersions", in $\mathbb{R}^3$, of the gluing spaces given by schematic diagrams presented in class.

**Answer to E9.0. Diagram 1:**


$X = [0, 1]$ with the points $0$ and $1$ glued together.

In more detail, the equivalence relation $\sim$ on $X = [0, 1]$ is defined by: $0 \sim 1$ so that $[0] = [1] = \{0, 1\}$; other equivalence classes are singletons, i.e., $[x] = \{x\}$ $\forall x \in X \setminus \{0, 1\}$.




*Could $[0, 1]$ with $0$ and $1$ glued together look like this?*

Informally, if we try to bend the interval and glue together its endpoints (we need a 2-dimensional to do this!), we get something like a loop. We conjecture that $X/{\sim}$ is homeomorphic to the circle. Let us prove this.

**Claim 1:** If $X = [0, 1]$ and $\sim$ is the equivalence relation described above ("gluing together $0$ and $1$"), then the quotient space $X/{\sim}$ is homeomorphic to the circle $S^1$.

**Proof:** a homeomorphism $X/{\sim} \to S^1$ is, in particular, a continuous map. All such continuous maps are described by the universal mapping property for a quotient space: namely, they are the same as continuous maps $f\colon X \to S^1$ such that $f(0) = f(1)$.

We view $S^1$ as the subset $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Put $f(t) = (\cos 2\pi t, \sin 2\pi t)$ so that $f(0) = f(1)$. Note that $f$ has distinct values on all $\sim$-equivalence classes so $f$ is injective on $X/\sim$. Clearly, $f$ is also surjective; $f$ is continuous because its components, $\cos 2\pi t$ and $\sin 2\pi t$, are continuous functions on $[0, 1]$.

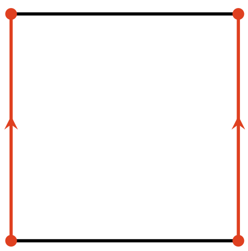
Thus, $f\colon X/\sim \to S^1$ is a **continuous bijection.** Note that:

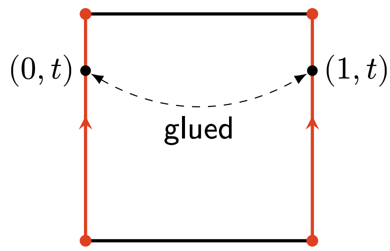
- $X = [0, 1]$ is compact by the Heine-Borel lemma.
- So, $X/\sim$ is **compact** as the image of $X$ under the (continuous) quotient map $q$.
- $S^1$ is **Hausdorff** as it is a metric space.

By Topological Inverse Function Theorem, a continuous bijection $f$ from a compact to a Hausdorff space is a **homeomorphism.** We have rigorously proved that the closed interval with its ends glued together is homeomorphic to a circle.

**Remark:** the space $[0, 1]/\sim$ that we have considered can be called "the abstract circle". We have thus embedded the abstract circle in $\mathbb{R}^2$.

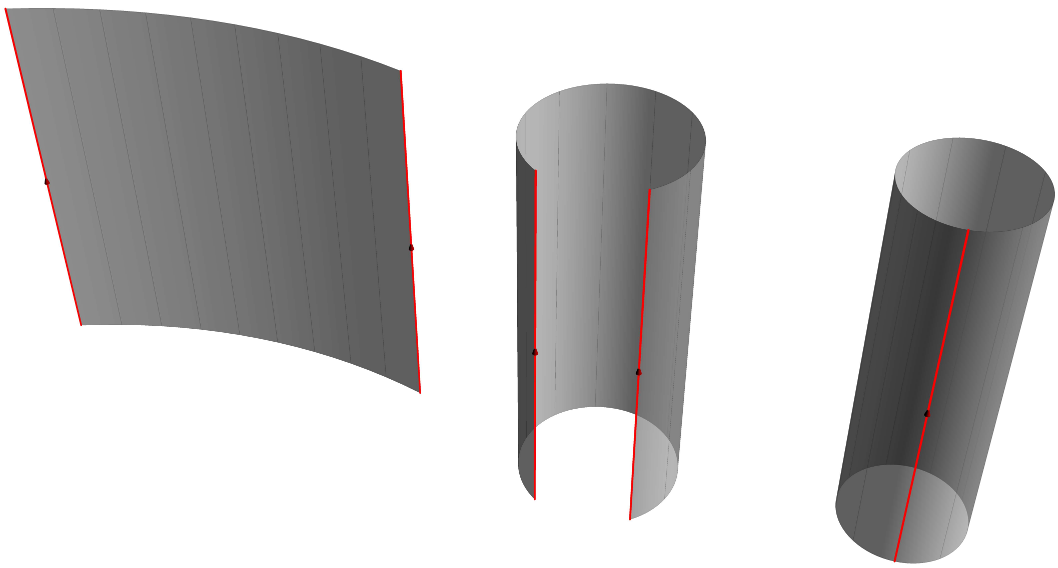
**Diagram 2:**

Diagram: 

In more detail: 

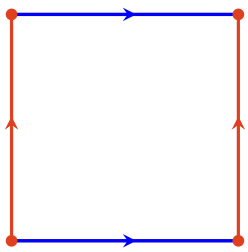

The diagram on the left indicates (using colour) that two parallel edges of the square $X = [0, 1] \times [0, 1]$ must be glued together. The arrows on the coloured edges, which point in the same direction, specify which point is glued to which: namely, a point at distance $t$ from the bottom left corner is identified with the point at the same distance $t$ from the bottom right corner. This is shown in more detail in the diagram on the right.

We try to construct the quotient space informally by bending the square and gluing the edges in $\mathbb{R}^3$. The figure shows the process of bending the square and gluing the opposite sides to obtain a **cylinder.** It is also easy to construct a homeomorphism from $X/\sim$ to the cylinder: writing the cylinder as $S^1 \times [0,1] \subseteq \mathbb{R}^3$ and put $f(t,u) = (\cos 2\pi t, \sin 2\pi t, u)$ where $(t,u) \in [0,1] \times [0,1]$. We omit the proof that $f$ is a homeomorphism, which is similar to Diagram 1. Thus, Diagram 2 defines a topological space $X/\sim$ which can be called "the abstract cylinder", and we have just embedded this space in $\mathbb{R}^3$.



*Bending the square in $\mathbb{R}^3$ to glue its opposite edges together and obtain a cylinder* [Link to online interactive 3D diagram]

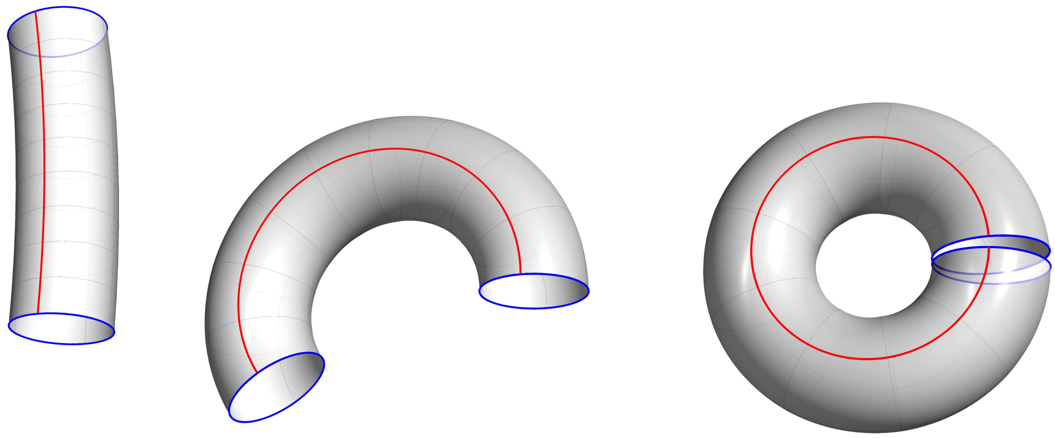
**Diagram 3:**



Here we glue together the points $(t,0)$ and $(t,1)$ for all $t \in [0,1]$, and glue together $(0,u)$ with $(1,u)$ for all $u \in [0,1]$, of the square $X = [0,1] \times [0,1]$. The resulting quotient space

$X/\sim$ may be called a "schematic torus": it is not difficult to show that it is homeomorphic to $\mathbb{T}^2 = S^1 \times S^1$.

Informally, the torus embedded in $\mathbb{R}^3$ can be obtained by stretching and bending the cylinder obtained above in order to glue the two circular edges together, see Figure.



*Bending the cylinder to glue its circular edges together and obtain a torus* [Link to online interactive 3D diagram]

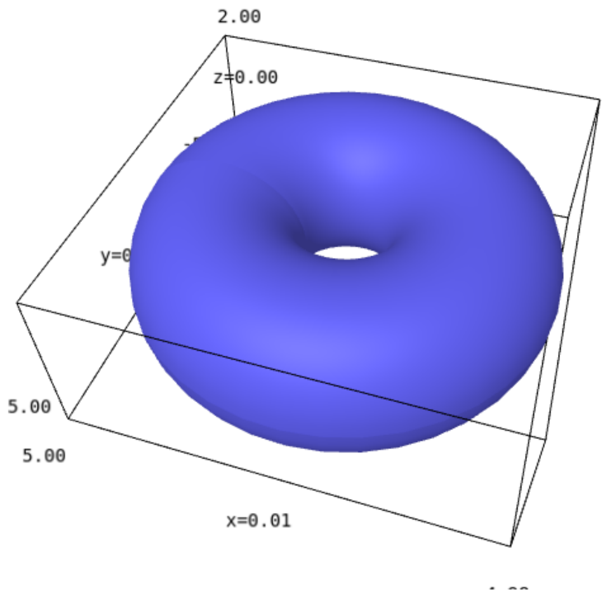
We will now write down an embedding $f$ of $X/\sim$ in $\mathbb{R}^3$, providing explicit formulas for the embedding of $\mathbb{T}^2$ in $\mathbb{R}^3$ described in the lectures.

Points $(t, 0) \in X$ are mapped onto the circle of radius $r$ centred at $(R, 0, 0)$ in the $xz$ plane: $f((t, 0)) = (R + r\cos 2\pi t, 0, r\sin 2\pi t)$.

Now, $f((t, u))$ is defined as $f((t, 0))$ rotated around the $z$ axis through the angle of $2\pi u$. Recall that the matrix of such rotation is $Rot_z(2\pi u) = \begin{pmatrix} \cos 2\pi u & -\sin 2\pi u & 0 \\ \sin 2\pi u & \cos 2\pi u & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We thus have

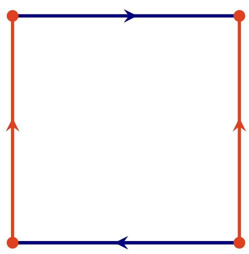
$$\begin{aligned} f((t, u)) &= Rot_z(2\pi u) f((t, 0)) \\ &= ((R + r\cos 2\pi t)\cos 2\pi u, (R + r\cos 2\pi t)\sin 2\pi u, r\sin 2\pi t). \end{aligned}$$

The function $f$ "respects" gluing, i.e., takes the same value on $(t, 0)$ and $(t, 1)$, similarly for $(0, u)$ and $(1, u)$. Hence $f$ gives a well-defined function $X/\sim \to \mathbb{R}^3$. As long as $R > r$, the function $f$ is injective on $X/\sim$, which is compact. Applying the Topological Inverse Function Theorem as we did earlier, we conclude that $f$ is an embedding.




*The image of $f$ is the familiar "surface of a doughnut"* [Link to online interactive 3D diagram]

**Diagram 4:**



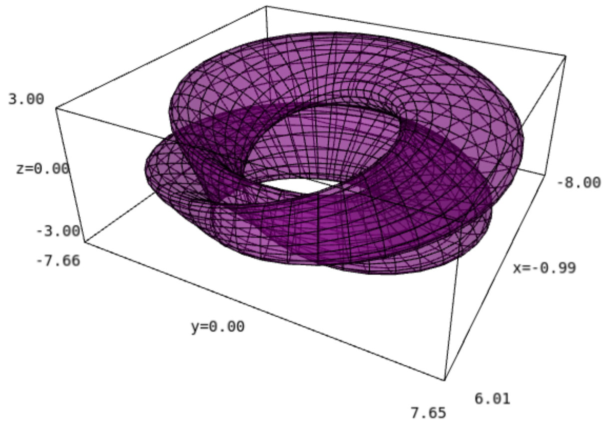
Attention! The arrows tell us that on the horizontal edges, we glue together the points $(t, 0)$ and $(1 - t, 1)$ for all $t \in [0, 1]$. The quotient space $X/\sim$ is the "schematic Klein bottle". One can prove (which is beyond the scope of our course) that $X/\sim$ is **not embeddable** in $\mathbb{R}^3$.

One can construct a function $f\colon X \to \mathbb{R}^3$ which glues together the points identified by $\sim$ but also some other points. The resulting continuous map $X/{\sim} \to \mathbb{R}^3$ is usually called an immersion.

An explicit formula for $f\colon X \to \mathbb{R}^3$ can be as follows: writing $\varphi = 2\pi t$, $\theta = 2\pi u$,

$$\begin{aligned} f(t,u) = \Big( &(R + \cos\frac{\theta}{2}\sin\varphi - \sin\frac{\theta}{2}\sin 2\varphi)\cos\theta, \\ &(R + \cos\frac{\theta}{2}\sin\varphi - \sin\frac{\theta}{2}\sin 2\varphi)\sin\theta, \\ &\sin\frac{\theta}{2}\sin\varphi + \cos\frac{\theta}{2}\sin 2\varphi\Big). \end{aligned}$$

It is easy to verify that $f$ respects gluing: $f(0,u) = (R\cos\theta, R\sin\theta, 0) = f(1,u)$ and also $f(t,0) = (R + \sin\varphi, 0, \sin 2\varphi)$ equals $f(1-t,1) = (R - \sin(2\pi - \varphi), 0, -\sin(4\pi - 2\varphi))$. However, $f$ also glues the line $t = \frac{1}{2}$ to the line $t = 0$ (and $t = 1$). The resulting 3d surface, a "symmetric" immersion of the Klein bottle in $\mathbb{R}^3$, is as shown in the Figure.




[Link to online interactive 3D diagram]

## References for the exercise sheet

Quotient spaces and surfaces are discussed in [Sutherland, Chapter 15], which includes proofs of the theoretical results given above and the embedding of the torus in $\mathbb{R}^3$ which we considered here. The Klein bottle immersions in $\mathbb{R}^3$ are discussed in popular topology resources (see this example).

The images of the “surface of a doughnut” and the “bagel-like” Klein bottle were generated by 3d plotting in SageMath computer algebra system. The SageMath code for the Klein bottle was generated by OpenAI ChatGPT.