Week 9

Topological properties of product spaces

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The universal mapping property of the product space $X \times Y$

Any function $f\colon Z \to X \times Y$ must output a point of $X \times Y$ which is a pair, hence must be of the form $f(z) = (f_X(z), f_Y(z))$. Formally, the "components" of f are

$$f_X : Z \xrightarrow{f} X \times Y \xrightarrow{p_X} X, \quad f_Y : Z \xrightarrow{f} X \times Y \xrightarrow{p_Y} Y$$

where p_X, p_Y are the projections. The next result is called **the universal mapping property** because it describes **all** possible continuous functions with the codomain $X \times Y$.

Theorem 9.1: the universal mapping property.

Let X, Y, Z be topological spaces. There is a 1-to-1 correspondence between

- continuous functions $f\colon Z \to X \times Y$, and
- pairs of continuous functions $(f_X\colon Z \to X, f_Y\colon Z \to Y)$.

To $f\colon Z \to X \times Y$ there corresponds the pair $(f_X = p_X \circ f, f_Y = p_Y \circ f)$. Vice versa, to a pair (f_X, f_Y) there corresponds the function f defined by $f(z) = (f_X(z), f_Y(z))$.

Remark: put simply, the Theorem says that $f\colon Z \to X \times Y$ is continuous **if and only if** the functions $f_X = p_X \circ f$, $f_Y = p_Y \circ f$ are continuous.

Proof of the Theorem. Let $f\colon Z \to X \times Y$ be a continuous function. Since p_X, p_Y are continuous by Proposition 8.1, and a composition of continuous functions is continuous, the functions $f_X = p_X \circ f$, $f_Y = p_Y \circ f$ are continuous.

Vice versa, let pair $f_X\colon Z \to X$, $f_Y\colon Z \to Y$ be continuous functions. We need to check that the function $f = (f_X, f_Y)$ is continuous. By definition of the product topology, an arbitrary open subset of $X \times Y$ is a union $\bigcup_{\alpha\in I} U_\alpha \times V_\alpha$ of open rectangles, where, for each $\alpha \in I$, U_α is open in X and V_α is open in Y. The preimage of a union is the union of preimages, so

$$f^{-1}\Big(\bigcup_{\alpha\in I} U_\alpha \times V_\alpha\Big) = \bigcup_{\alpha\in I} f^{-1}(U_\alpha \times V_\alpha).$$

Note that $f^{-1}(U_\alpha \times V_\alpha)$ consists of $z \in Z$ such that $f_X(z) \in U_\alpha$ and $f_Y(z) \in V_\alpha$. In other words, $f^{-1}(U_\alpha \times V_\alpha) = f_X^{-1}(U_\alpha) \cap f_Y^{-1}(V_\alpha)$ which is open in Z, because $f_X^{-1}(U_\alpha)$ and $f_Y^{-1}(V_\alpha)$ are preimages of open sets under the confinuous functions f_X, f_Y, and the intersection of two open sets is open.

Therefore, $\bigcup_{\alpha\in I} f^{-1}(U_\alpha \times V_\alpha)$ is open as a union of open sets. We have verified the definition of "continuous" for f.

It remains to note that, given $f\colon Z \to X \times Y$, taking $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$ then constructing (f_X, f_Y) brings us back to the function f. Also, given the functions f_X, f_Y, if $f = (f_X, f_Y)$ then taking $p_X \circ f$ and $p_Y \circ f$ returns us to the functions f_X and f_Y. This shows that we have two mutually inverse correspondences between functions $Z \to X \times Y$ and pairs of functions $Z \to X$, $Z \to Y$. Hence we have a bijective (1-to-1) correspondence between the two sets of functions, as claimed. $\square$

Topological properties of the product space

We would like to understand how the topological properties of the product space $X \times Y$ are determined by the topological properties of the spaces X and Y. The following result helps us by showing how homeomorphic copies of the space X sit inside $X \times Y$:

Lemma 9.2: embedding of X in $X \times Y$.

For each $y_0 \in Y$, the subspace $X \times \{y_0\}$ of $X \times Y$ is homeomorphic to X.

Proof. Consider the embedding map $i_{y_0} : X \to X \times Y$, $x \to (x, y_0)$. Note that $p_X \circ i_{y_0}$ is the identity map $x \mapsto x$ of X which is continuous, and $p_Y \circ i_{y_0}$ is the constant map $\mathrm{const}_{y_0} : X \to Y$, which is continuous. Hence by the Universal Mapping Property, Theorem 9.1, i_{y_0} is a continuous map. We can restrict the codomain and consider i_{y_0} as a continuous map $X \to X \times \{y_0\}$.

The projection $p_X : X \times Y \to X$ is continuous by Proposition 8.1, so its restriction $p_X|_{X\times\{y_0\}}$ is continuous.

The composition $x \mapsto (x, y_0) \mapsto x$ shows that $p_X|_{X\times\{y_0\}} \circ i_{y_0} = \mathrm{id}_X$. Also, the composition $(x, y_0) \mapsto x \mapsto (x, y_0)$ shows that $i_{y_0} \circ p_X|_{X\times\{y_0\}} = \mathrm{id}_{X\times\{y_0\}}$. Therefore, i_{y_0} and $p_X|_{X\times\{y_0\}}$ are two mutually inverse continuous maps. We have verified the definition of "homeomorphism" for $i_{y_0} : X \to X \times \{y_0\}$. □

Remark: in the same way, if $x_0 \in X$, the subspace $\{x_0\} \times Y$ of $X \times Y$ is homeomorphic to Y.

Proposition 9.3: Hausdorfness and connectedness of the product.

If X, Y are non-empty topological spaces, then

(a) X and Y are both Hausdorff $\iff$ $X \times Y$ is Hausdorff;

(b) X and Y are both connected $\iff$ $X \times Y$ is connected.

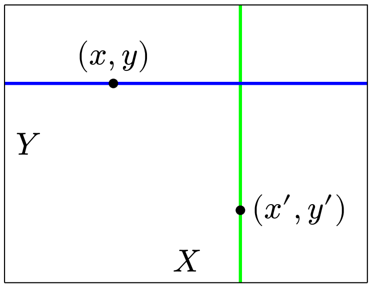

Figure 9.1: *if X, Y are connected, any two points of $X \times Y$ are joined by a connected set*

Proof (not given in class). (a) $\Rightarrow$: assume X, Y are Hausdorff, and let $(x, y) \neq (x', y')$ be two distinct points of $X \times Y$. Then either $x \neq x'$, or $y \neq y'$, or both.

If $x \neq x'$, take U, U' to be open sets in X such that $x \in U$, $x' \in U'$ and $U \cap U' = \emptyset$. Then the points (x, y) and (x', y') lie in disjoint open cylinder sets $U \times Y$ and $U' \times Y$. If $y \neq y'$, a similar argument leads to disjoint open cylinders $X \times V$ and $X \times V'$. We have thus verified the definition of "Hausdorff" for $X \times Y$.

$\Leftarrow$: assume $X \times Y$ is Hausdorff. Pick a point y_0 in the non-empty space Y. Subspaces of a Hausdorff space are Hausdorff (Proposition 3.3), so $X \times \{y_0\}$ is Hausdorff; it is homeomorphic to X by Lemma 9.2, and Hausdorffness is a topological property (Proposition 3.1), so X is Hausdorff. Similarly, Y is Hausdorff.

(b) $\Rightarrow$: assume X, Y are connected. Any two points (x, y) and (x', y') of $X \times Y$ lie in the set $(X \times \{y\}) \cup (\{x'\} \times Y)$, see Figure 9.1. Connectedness is a topological property, so $X \times \{y\}$, which is homeomorphic to X by Lemma 9.2, is connected. Similarly, $\{x'\} \times Y$ is connected. The union of two connected sets, which have a common point, is connected by Lemma 7.3, so (x, y) and (x', y') lie in the same connected component of $X \times Y$. Since the two points were arbitrary, $X \times Y$ has only one connected component, i.e., is connected.

$\Leftarrow$: if $X \times Y$ is connected, then $X = p_X(X \times Y)$ and $Y = p_Y(X \times Y)$ are connected, because p_X, p_Y are continuous (Proposition 8.1) and a continuous image of a connected space is connected (Theorem 7.1). □

The baby Tychonoff theorem about compactness of $X \times Y$

We will now deal with compactness of $X \times Y$ which is more intricate than Hausdorfness and connectedness. Since the product topology on $X \times Y$ is given by its base $\mathscr{B}$, we first prove a lemma which allows us to check compactness using only covers by sets from $\mathscr{B}$.

> **Lemma 9.4: a "basic compact" is a compact.**
>
> Let $\mathscr{B}$ be a base of topology on a space X. Suppose that every basic cover of X (i.e., a cover by sets from $\mathscr{B}$) has a finite subcover. Then X is compact.

Proof. By definition of a base, every open set $U \subseteq X$ can be written as a union of basic open sets (i.e., members of $\mathscr{B}$). Call these basic open sets "the children" of U.

Suppose that $\mathscr{C}$ is a cover of X by open sets U_α, $\alpha \in I$. Consider the collection $\mathscr{C}_1 = \{$all children of all sets from $\mathscr{C}\}$. Each set in $\mathscr{C}$ is the union of its children, so $\bigcup \mathscr{C}_1 = \bigcup \mathscr{C} = X$.

Thus, $\mathscr{C}_1$ is a basic open cover of X. Then by assumption, a finite cover $V_1, \dots, V_n$ can be chosen from $\mathscr{C}_1$: $V_1 \cup \dots \cup V_n = X$. The sets $V_1, \dots, V_n$ may not lie in $\mathscr{C}$, but their "parents" do. Consider the finite subcollection of $\mathscr{C}$ given by

$$\text{a parent of } V_1,\ \text{a parent of } V_2,\ \dots,\ \text{a parent of } V_n.$$

Here "a parent" means an arbitrary choice of parent if V_i has more than one parent. Since $V_i \subseteq$ (a parent of V_i), the union of the n "parents" is also X. Given any open cover $\mathscr{C}$ of X, we constructed a finite subcover, thus verifying the definition of "compact" for X. □

> **Theorem 9.5: the baby Tychonoff theorem.**
>
> If X, Y are non-empty spaces, both X and Y are compact $\iff$ $X \times Y$ is compact.

Proof. $\Leftarrow$: $X \times Y$ is compact, so $X = p_X(X \times Y)$ is compact by Theorem 4.2 (continuous image of compact) as p_X is continuous by Proposition 8.1. Similarly, Y is compact.

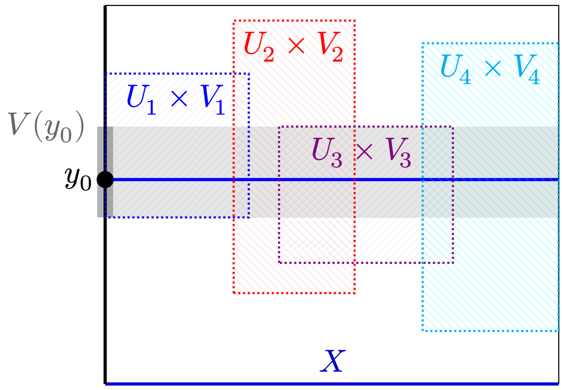

Figure 9.2: *if finitely many open rectangles cover* $X \times \{y_0\}$, *they cover an open cylinder* $X \times V(y_0)$

$\Rightarrow$: let X, Y be compact. We assume that $\mathscr{C}$ is **a cover of** $X \times Y$ **by open rectangles** and show that $\mathscr{C}$ has a finite subcover. Since, by definition, open rectangles form a base of the topology on $X \times Y$, Lemma 9.4 will imply that $X \times Y$ is compact.

For each $y_0 \in Y$, $X \times \{y_0\}$ is homeomorphic to X (Lemma 9.2), hence $X \times \{y_0\}$ **is a compact set,** which by compactness criterion 4.1 is covered by a finite subcollection of open rectangles $U_1 \times V_1, \ldots, U_n \times V_n$ from $\mathscr{C}$. We may assume that each $U_i \times V_i$ intersects $X \times \{y_0\}$ (otherwise delete it from the finite cover), so $V_i \ni y_0$ for all $i = 1, \ldots, n$.

Define the open neighbourhood $V(y_0)$ to be $V_1 \cap \cdots \cap V_n$. Then each open rectangle $U_i \times V_i$ contains $U_i \times V(y_0)$, and so the open rectangles $U_1 \times V_1, \ldots, U_n \times V_n$, which form the finite cover of $X \times \{y_0\}$, **also cover the open cylinder** $X \times V(y_0)$, see Figure 9.2.

We have thus constructed an open neighbourhood $V(y_0)$ for **every point** y_0 **of** Y**.** We now use **compactness of** Y to choose a finite subcover of Y by these neighbourhoods: say, $V(y_1), \ldots, V(y_m)$ such that $V(y_1) \cup \cdots \cup V(y_m) = Y$.

Then the union of the cylinders $X \times V(y_1), \ldots, X \times V(y_m)$ is $X \times Y$. Also, by the above construction, each cylinder $X \times V(y_j)$ is covered by a finite subcollection of $\mathscr{C}$. The union of these m finite subcollections is a **finite subcover, chosen from** $\mathscr{C}$**, for the whole of** $X \times Y$, as required. $\square$

The following corollary is now easily obtained by induction. We understand the n-fold product $X_1 \times X_2 \times \dots \times X_n$ as the iterated product $(((X_1 \times X_2) \times X_3) \times \dots) \times X_n$.

Corollary: the product of finitely many compact spaces is compact.

If $X_1, \dots, X_n$ are compact topological spaces, then the product space $X_1 \times \dots \times X_n$ is compact. □

The Heine-Borel Theorem

The baby Tychonoff theorem is now used to extend the Heine-Borel Lemma, Theorem 5.2, which says that $[0, 1]$ is compact, to a result which describes all compact sets in Euclidean spaces $\mathbb{R}^n$.

Theorem 9.6: The Heine-Borel Theorem.

In the Euclidean space $\mathbb{R}^n$, a set K is compact iff K is closed and bounded.

Proof. The "only if" part is immediate by Proposition 5.1: if K is a compact set in any metric space, then K is closed and bounded.

To prove the "if" part, assume that K is a closed bounded subset of $\mathbb{R}^n$. Any bounded set is a subset of an n-dimensional cube $\{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| \leq M \ \forall i = 1, \dots, n\}$, for some $M > 0$. This cube of side length $2M$ is the product space $[-M, M] \times \dots \times [-M, M] = [-M, M]^n$.

Note that the closed bounded interval $[-M, M] \subseteq \mathbb{R}$ is homeomorphic to $[0, 1]$ (a homeomorphism is afforded, for example, by a linear function mapping $[0, 1]$ onto $[-M, M]$). By the Heine-Borel Lemma, $[0, 1]$ is compact, and so $[-M, M]$ is also compact. By baby Tychonoff theorem 9.5 and its Corollary, $[-M, M]^n$ is compact.

Thus K is a closed subset of the compact $[-M, M]^n$, and so by Proposition 4.3, K is compact. □

The torus and its embedding in $\mathbb{R}^3$

The theory that we have developed so far can be used to construct embeddings of abstractly defined topological spaces in a Euclidean space. By an embedding, we mean the following:

Definition: embedding.

Let X, Y be topological spaces. An **embedding of X in Y** is a map $f\colon X \to Y$ such that restricting the codomain gives a homeomorphism $f\colon X \xrightarrow{\sim} f(X)$.

In other words, an embedding means constructing inside Y a subspace homeomorphic to X. This is important in many applications.

Example: embedding of the torus in $\mathbb{R}^3$.

Let S^1 denote the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in the Euclidean plane. Define the **torus** as the product space

$$\mathbb{T}^2 = S^1 \times S^1.$$

Show: $\mathbb{T}^2$ is compact. Construct an embedding of $\mathbb{T}^2$ in the Euclidean space $\mathbb{R}^3$.

Solution: S^1 is closed and bounded in $\mathbb{R}^2$, hence compact by Heine-Borel Theorem 9.6.

(There are alternative ways to show compactness of S^1; for example, S^1 is the image of the map $[0, 1] \to \mathbb{R}^2$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ which is continuous; $[0, 1]$ is compact, and a continuous image of a compact is compact.)

It follows that $\mathbb{T}^2 = S^1 \times S^1$ is compact by baby Tychonoff theorem 9.5.

It is not automatic that an embedding of $\mathbb{T}^2$ in $\mathbb{R}^3$ must exist: note that S^1 is a subspace of $\mathbb{R}^2$, so $\mathbb{T}^2$ is naturally a subspace of $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ and not $\mathbb{R}^3$. Yet we can construct an **injective** function $f\colon S^1 \times S^1 \to \mathbb{R}^3$, as follows.

A point of $\mathbb{T}^2$ is a pair (P, Q) where P is a point on the first circle, and Q is a point on the second unit circle in $S^1 \times S^1$. It is convenient to represent the two points by their

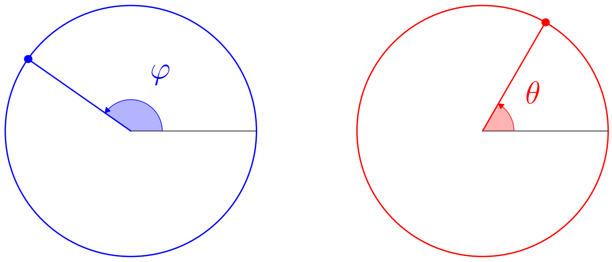

Figure 9.3: *the torus* $\mathbb{T}^2$ *is defined as the product of two circles, so a point of* $\mathbb{T}^2$ *is a pair of circle points and is represented by a pair of angles* (φ, θ)

angle coordinates (φ, θ) with $\varphi, \theta \in [0, 2\pi)$, see Figure 9.3.

Fix two radii R, r such that $R > r > 0$. Informally, we will think of the first circle in $S^1 \times S^1$ (the blue circle in Figure 9.3) as the circle of radius R in the horizontal xy plane in $\mathbb{R}^3$. The torus $\mathbb{T}^2$ is the disjoint union

$$\mathbb{T}^2 = \bigcup_{P \in S^1} \{P\} \times S^1,$$

and for each P on the blue circle, we map the subset $\{P\} \times S^1$ of the torus to the red circle of radius R, centred at P and orthogonal to the blue circle. Thus, $f\colon S^1 \times S^1 \to \mathbb{R}^3$ is given by

$$(\varphi, \theta) \mapsto \mathrm{Rotate}^{\varphi}_{z\text{-axis}}((R, 0, 0) + (\cos\theta, 0, \sin\theta))$$

as shown in Figure 9.4. An explicit formula for $f(\varphi, \theta)$ will be worked out in the tutorial.

Explanation why f is injective: f maps two points (φ, θ) and (φ', θ') on $\mathbb{T}^2$ such that $\varphi \neq \varphi'$, onto two disjoint red circles in Figure 9.4, to the f-images are distinct. The red circles are disjoint because we chose $r < R$. In the case $\varphi = \varphi'$, the two points are mapped by f onto the same red circle but to different points of the circle, as long as $\theta \neq \theta'$.

Explanation why f is continuous: the x-component f_x of f can be written as an algebraic expression in $\cos\varphi$, $\sin\varphi$, $\cos\theta$ and $\sin\theta$ (see the formula given in the tutorial). Note that $\cos\varphi$ and $\sin\varphi$ are the actual coordinates of the point on the first (blue) circle, hence they are continuous functions on $\mathbb{T}^2$ by Proposition 8.1. Same can be said of $\cos\theta$ and $\sin\theta$.

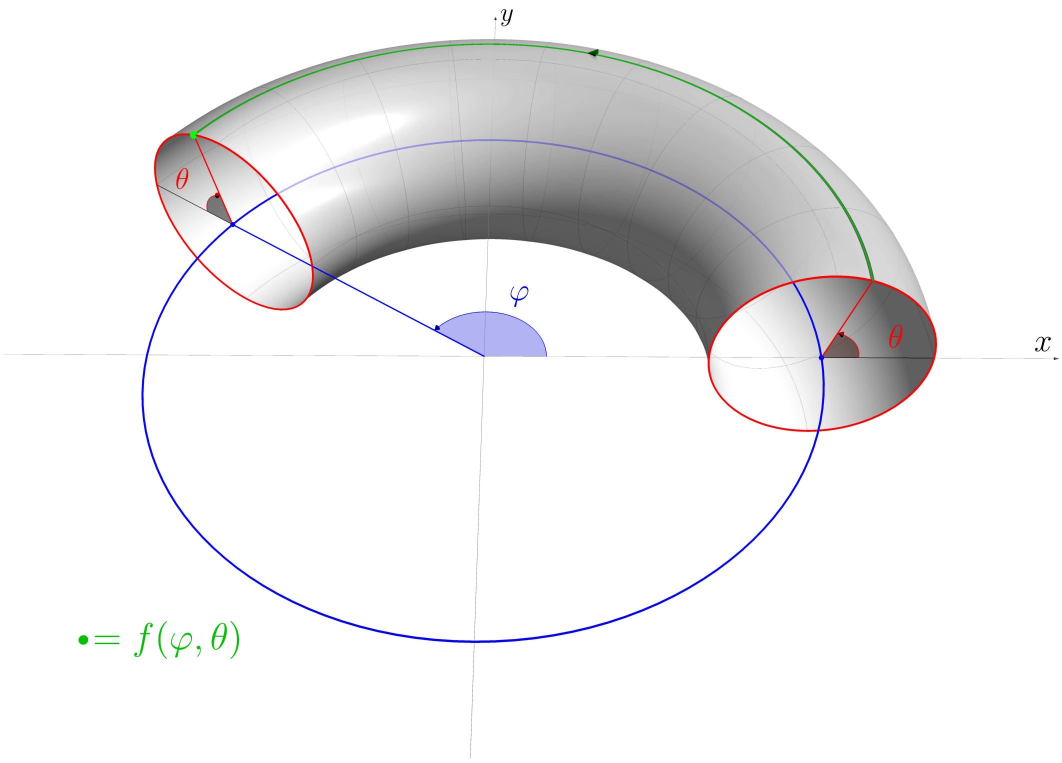

Figure 9.4: *the image of a point* $(\varphi, \theta) \in \mathbb{T}^2$ *in* $\mathbb{R}^3$ *is obtained by rotating, through* φ *around the* z *axis, the point* θ *on the red* xz *circle of radius* r *around* $(R, 0, 0)$. [Link to online interactive 3D diagram]

Sums and products of continuous functions are continuous (this is known to be true for metric spaces; for general topological spaces, see E4.5). Hence f_x is a continuous function from $\mathbb{T}^2$ to $\mathbb{R}$.

In the same way one shows that f_y and f_z are continuous $\mathbb{R}$-valued functions on $\mathbb{T}^2$. Hence $f\colon \mathbb{T}^2 \to \mathbb{R}^3$ is continuous by the Universal Mapping Property, Theorem 9.1.

Proof that f is an embedding: restricting the codomain of continuous injection f gives the the continuous bijection $f\colon \mathbb{T}^2 \to f(\mathbb{T}^2)$. We **do not need continuity of the inverse map:** as $\mathbb{T}^2$ is compact and $f(\mathbb{T}^2)$ is metric hence Hausdorff, by the Topological Inverse Function Theorem 4.5 $f\colon \mathbb{T}^2 \xrightarrow{\sim} f(\mathbb{T}^2)$ is a homeomorphism. □

The Tychonoff theorem (not done in class, not examinable)

Baby Tychonoff theorem 9.5 is a particular case of a result that we state here for completeness. The proof is beyond the scope of this course and can be found in the literature.

Let $(X_\alpha)_{\alpha \in I}$ be a collection of topological spaces. The topology on the Cartesian product $\prod_{\alpha \in I} X_\alpha$ is defined to have base $\mathscr{B}$ of sets of the form $\prod_{\alpha \in I} U_\alpha$ where (1) $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$; (2) U_α is open in X_α for all $\alpha \in I$.

The product topology on $X \times Y$ is a particular case: (1) can be omitted if I is finite.

Theorem 9.7: the Tychonoff theorem.

Suppose the space X_α is not empty for all $\alpha \in I$. Then the product space $\prod_{\alpha \in I} X_\alpha$, defined above, is compact if, and only if, X_α is compact for all $\alpha \in I$. □

References for the week 9 notes

Theorem 9.1, the Universal Mapping Property of $X \times Y$, is [Sutherland, Proposition 10.11] as well as [Armstrong, Theorem (3.13)].

Lemma 9.2 about a homeomorphic copy $X \times \{y_0\}$ of X in $X \times Y$ is a strengthening of [Sutherland, Proposition 10.14] which only asserts that the map $i_{y_0} : x \mapsto (x, y_0)$ is continuous.

Proposition 9.3: (a) Hausdorffness of $X \times Y$ is [Sutherland, Proposition 11.17b], [Armstrong, Theorem (3.14)]; (b) connectedness is [Sutherland, Theorem 12.18], [Armstrong, Theorem (3.26)].

The baby Tychonoff theorem 9.5 is [Sutherland, Theorem 13.21] and [Armstrong, Theorem (3.15)]. Our proof follows [Armstrong]. In particular, our Lemma 9.4 is [Armstrong, Lemma (3.16)].

The Heine-Borel theorem 9.6 is [Sutherland, Theorem 13.22] and [Armstrong, Theorem (3.1)].

The general Tychonoff theorem 9.7 is not usually proved in introductory-level Topology textbooks. A proof can be found in [Willard, Theorem 17.8].