Week 8

Exercises (answers at end)

Version 2024/11/20. To accessible online version of these exercises

Exercise 8.0. This is an unseen exercise on closure, boundary and dense sets. Consider the sets $A = \{0, 1\} \subset \mathbb{R}$ and $B = \mathbb{R} \setminus A = (-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ as a subsets of four different topological spaces, given in the table below. Complete the table.

	The space X			
	$(\mathbb{R}$, antidiscrete$)$	$(\mathbb{R}$, cofinite$)$	$(\mathbb{R}$, Euclidean$)$	$(\mathbb{R}$, discrete$)$
$\overline{A}$ (closure in X)				
Is A dense in X? (yes/no)				
$\overline{B}$				
Is B dense in X? (yes/no)				
∂A				

Hint. You may wish to recall that $\overline{A}$ = the smallest closed set in X which contains A = $\{z \in X : \text{all open neighbourhoods of } z \text{ meet } A\}$ and that $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

Exercise 8.1. (a) Use the following two results,

- *a connected component of a topological space is a connected set* (Lemma 7.4),
- *if the space X has a connected dense subset then X is connected* (Lemma 7.11),

to show that each connected component of a topological space is a closed set.

(b) Deduce from (a) that if a topological space X has finitely many connected components, then each connected component is both closed and open in X.

(c) Give an example of a topological space where connected components are closed but not open.

Exercise 8.2. (a) Suppose that X is a topological space, points $x, y \in X$ are joined by a path in X, and points $y, z \in X$ are also joined by a path in X. Show that x, z are joined by a path in X.

(b) Furthermore, show that "$x \sim y \iff x, y$ are joined by a path in X" is an equivalence relation on X.

Equivalence classes defined by the relation $\sim$ from (b) are called **path-connected components of X.** In general, a path-connected component does not need to be open or closed in X. Nevertheless:

(c) Show that if X is an **open** subset of a **Euclidean space** $\mathbb{R}^n$, then each path-connected component of X is open. Deduce that an open connected subset of $\mathbb{R}^n$ is path-connected.

Week 8

Exercises — solutions

Version 2024/11/20. To accessible online version of these exercises

Exercise 8.0. This is an unseen exercise on closure, boundary and dense sets. Consider the sets $A = \{0,1\} \subset \mathbb{R}$ and $B = \mathbb{R} \setminus A = (-\infty, 0) \cup (0,1) \cup (1,+\infty)$ as a subsets of four different topological spaces, given in the table below. Complete the table.

	The space X			
	$(\mathbb{R}$, antidiscrete$)$	$(\mathbb{R}$, cofinite$)$	$(\mathbb{R}$, Euclidean$)$	$(\mathbb{R}$, discrete$)$
$\overline{A}$ (closure in X)	$\mathbb{R}$	A	A	A
Is A dense in X? (yes/no)	yes	no	no	no
$\overline{B}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	B
Is B dense in X? (yes/no)	yes	yes	yes	no
∂A	$\mathbb{R}$	A	A	$\emptyset$

Hint. You may wish to recall that $\overline{A} =$ the smallest closed set in X which contains $A = \{z \in X : \text{all open neighbourhoods of } z \text{ meet } A\}$ and that $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

Answer to E8.0. Explanation of the above entries in the table: let X be ($\mathbb{R}$, antidiscrete topology). The only closed sets in X are $\emptyset$ and $\mathbb{R}$. Of these, only $\mathbb{R}$ contains A. Hence $\overline{A}$, which is the smalest closed set containing A, is $\mathbb{R}$. In the same way $\overline{B} = \mathbb{R}$. Since $\mathbb{R} \setminus A = B$, we have $\partial A = \overline{A} \cap \overline{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

Now let X be ($\mathbb{R}$, cofinite topology). Note that the closed sets in the cofinite topology are finite sets and $\mathbb{R}$. Since A is finite, A is closed and $\overline{A} = A$. Since B is infinite, the smallest closed set which contains B is $\mathbb{R}$, hence $\overline{B} = \mathbb{R}$. We have $\partial A = A \cap \mathbb{R} = A$.

Next, let X be ($\mathbb{R}$, Euclidean topology); this is a Hausdorff space, so singletons $\{0\}$ and $\{1\}$ are closed, and $A = \{0\} \cup \{1\}$ is closed as a finite union of closed sets. Hence $\overline{A} = A$. Every open neighbourhood of 0 contains points from B, because an open set cannot consists just of points of the finite set A; hence $0 \in \overline{B}$. Similarly, $1 \in \overline{B}$. Hence $\mathbb{R} = \{0\} \cup \{1\} \cup B \subseteq \overline{B}$. We thus have $\overline{B} = \mathbb{R}$, and $\partial A = A \cap \mathbb{R} = A$.

Finally, if X is ($\mathbb{R}$, discrete topology), then every subset of X is closed and is equal to its own closure. So $\overline{A} = A$, $\overline{B} = B$ and $\partial A = A \cap B = \emptyset$.

In each case, a set is dense in $\mathbb{R}$ if its closure is $\mathbb{R}$.

Exercise 8.1. (a) Use the following two results,

- *a connected component of a topological space is a connected set* (Lemma 7.4),
- *if the space X has a connected dense subset then X is connected* (Lemma 7.11),

to show that each connected component of a topological space is a closed set.

(b) Deduce from (a) that if a topological space X has finitely many connected components, then each connected component is both closed and open in X.

(c) Give an example of a topological space where connected components are closed but not open.

Answer to E8.1. (a) Let $x \in X$, and denote the connected component of x by C. By Lemma 7.4, C is connected. By definition of "dense", C is dense in the subspace $\overline{C}$ of X. Therefore, by Lemma 7.11, $\overline{C}$ is connected. Since $\overline{C}$ contains x, this means that $\overline{C}$ must lies in the connected component of x, that is, $\overline{C} \subseteq C$. On the other hand, $C \subseteq \overline{C}$ for all sets C, see Claim 7.7. So $C = \overline{C}$, which is equivalent to C being closed.

(b) Assume that X is the union of finitely many connected components $C_1, \dots, C_n$. In part (a), we proved that $C_1, \dots, C_n$ are closed, and now we will show that C_1 is open. Recall that $C_1, \dots, C_n$ are equivalence classes, hence they are disjoint, and

$$C_1 = X \setminus (C_2 \cup \dots \cup C_n).$$

Finite unions of closed sets are closed, so $C_2 \cup \dots \cup C_n$ is closed, and its complement C_1 is open, as claimed.

(c) The set $\mathbb{Q}$ of rational numbers, viewed as the subspace of the Euclidean line $\mathbb{R}$, is totally disconnected, as shown in E7.1(3): every non-empty connected subset of $\mathbb{Q}$ is a singleton. Since connected components of $\mathbb{Q}$ are connected sets, it follows that **connected components of $\mathbb{Q}$ are singletons.** But a singleton (a one-point set) is **not open** in $\mathbb{Q}$: we showed exactly that in E7.1(2).

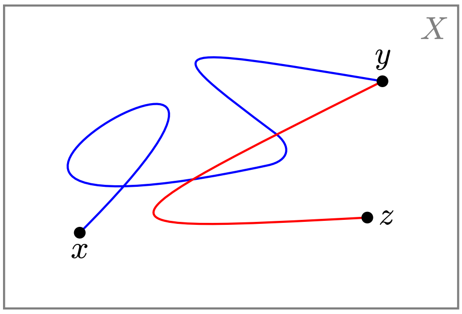
Exercise 8.2. (a) Suppose that X is a topological space, points $x, y \in X$ are joined by a path in X, and points $y, z \in X$ are also joined by a path in X. Show that x, z are joined by a path in X.

(b) Furthermore, show that "$x \sim y \iff x, y$ are joined by a path in X" is an equivalence relation on X.

Equivalence classes defined by the relation $\sim$ from (b) are called **path-connected components of X.** In general, a path-connected component does not need to be open or closed in X. Nevertheless:

(c) Show that if X is an **open** subset of a **Euclidean space** $\mathbb{R}^n$, then each path-connected component of X is open. Deduce that an open connected subset of $\mathbb{R}^n$ is path-connected.

Answer to E8.2. (a) Let ϕ be a path joining x and y in X. By definition, this means that $\phi\colon [0,1] \to X$ is a continuous function, $\phi(0) = x$ and $\phi(1) = y$. Similarly, a path joining y and z is a continuous function $\psi\colon [0,1] \to X$ with $\psi(0) = y$ and $\psi(1) = z$.



A path joining x and z will be the blue path from x to y followed by the red path from y to z; this is known as the concatenation of two paths

We need to construct a continuous function $\chi\colon [0,1] \to X$ such that $\chi(0) = x$ and $\chi(1) = z$. Intuitively, we will construct a path which will first trace the path from x to y, then trace the path from y to z; this is called the **concatenation of paths** in the Figure, this will look like the continuous curve which is the union of the blue curve from x to y and the red curve from y to z.

The functions $\hat{\phi}\colon [0,\frac{1}{2}] \to X$, $\hat{\phi}(t) = \phi(2t)$, and $\widehat{\psi}\colon [\frac{1}{2},1] \to X$, $\widehat{\psi}(t) = \psi(2t-1)$, are continuous as compositions of continuous functions. Define $\chi\colon [0,1] \to X$ by

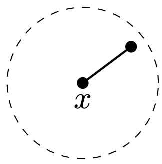
$$\chi(t) = \begin{cases} \hat{\phi}(t), & t \in [0,\frac{1}{2}], \\ \widehat{\psi}(t), & t \in [\frac{1}{2},1]. \end{cases}$$

Note that $\chi(\frac{1}{2})$ is well-defined as $\hat{\phi}(\frac{1}{2}) = \widehat{\psi}(\frac{1}{2}) = y$. Moreover, $\chi(0) = \phi(2\times 0) = x$ and $\chi(1) = \psi(2\times 1 - 1) = z$.

We prove that χ is continuous using the closed set criterion of continuity, Proposition 2.5. If $F \subseteq X$ is closed in X, $\chi^{-1}(F) = \hat{\phi}^{-1}(F) \cup \widehat{\psi}^{-1}(F)$. Since $\hat{\phi}$ is continuous, $\hat{\phi}^{-1}(F)$ is closed in $[0,\frac{1}{2}]$, hence compact. Similarly, $\widehat{\psi}^{-1}(F)$ is compact. A union of two compact sets is compact by E5.2, so $\chi^{-1}(X)$ is compact, hence closed in the Hausdorff space $[0,1]$. We have verified the closed set criterion of continuity for χ. Thus, x and z are joined by the path χ.

(b) Part (a) shows that the relation $\sim$ is transitive. We note that $\sim$ is reflexive: $x \sim x$ because the constant path, $\phi(t) = x$ for all $t \in [0, 1]$, joins x and x. Moreover, $\sim$ is symmetric: if ϕ is a path joining x and y, then the path $\bar{\phi}\colon [0, 1] \to X$, $\bar{\phi}(t) = \phi(t - 1)$, joins y and x. We conclude that $\sim$ is an equivalence relation.

(c) Suppose that $X \subseteq \mathbb{R}^n$ is an open set. If $x \in X$, take $r > 0$ such that $B_r(x) \subseteq X$. Note that in an open ball $B_r(x)$ in a Euclidean space, every point is joined to x by a path — in fact, by a straight line segment. Hence $B_r(x)$ lies inside the path-connected component of x.



In $\mathbb{R}^n$, any point of $B_r(x)$ can be joined by a straight-line path to x

Thus, the path-connected component of each point of X contains an open ball centred at that point. This is the definition of an open set in a metric space.

Now, if X is a **connected** open set in $\mathbb{R}^n$, then X cannot have more than one path-connected component: otherwise X would be a union of disjoint **open** path-connected components, and a union of disjoint non-empty open sets is disconnected. Thus, X consists of one path-connected component, and is path-connected.

References for the exercise sheet

E8.1(a), **a connected component is closed,** is [Willard, Theorem 26.12]. The example in E8.1(c) is both [Armstrong, Example 3 in Section 3.5] and [Willard, Example 26.13a].

E8.2(a) (concatenation of paths) is based on [Sutherland, Lemma 12.2]. E8.2(c), **a connected open set in Euclidean space is path-connected,** is [Sutherland, Proposition 12.25] and [Willard, Corollary 27.6].