Week 8

Definition of the product topology

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Writing $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, we may define many metrics on $\mathbb{R}^2$, for example d_1, d_2 and d_∞, see Figure 2.1. Yet, all these metrics define the same topology. This is not a coincidence: given topological spaces X, Y, we will now construct the standard topology $X \times Y$ called the product topology. Importantly, the construction extends to the Cartesian product of infinitely many topological spaces.

Key results of this chapter include the Tychonoff Theorem (only the baby version will be proved in class) and the Heine-Borel Theorem. We will consider one the topologists' favourite product space examples: the torus.

The Cartesian product

We begin with a reminder about the Cartesian product of sets.

> **Definition: Cartesian product of two sets.**
>
> Let X, Y be sets. The **Cartesian product** $X \times Y$ is the set of all pairs (x, y) with $x \in X$, $y \in Y$.

The Cartesian product construction extends to arbitrary finite or infinite collections of sets:

- the Cartesian product of n sets is a set of n-tuples,

$$X_1 \times \cdots \times X_n = \prod_{k=1}^{n} X_k = \{(x_1, \dots, x_n) : x_k \in X_k \; \forall k = 1, \dots, n\};$$

- for a sequence $X_1, X_2, \dots$ of sets, the Cartesian product is a set of sequences,

$$\prod_{k=1}^{\infty} X_k = \{(x_k)_{k \geq 1} : x_k \in X_k \; \forall k \geq 1\};$$

- for a collection $\{X_\alpha : \alpha \in I\}$ of sets, the Cartesian product is a set of collections of elements indexed by I,

$$\prod_{\alpha \in I} X_\alpha = \{(x_\alpha)_{\alpha \in I} : x_\alpha \in X_\alpha \; \forall \alpha \in I\}.$$

We will initially focus on the Cartesian product of two sets.

Subsets of $X \times Y$ of a special form will be important to us:

Definition: rectangle sets, cylinder sets.

A **rectangle set** in $X \times Y$ is a set of of the form $A \times B$ where $A \subseteq X$ and $B \subseteq Y$.

A **cylinder set** in $X \times Y$ is a rectangle set of the form $A \times Y$ or $X \times B$.

Figure 8.1 illustrates these types of subsets of $X \times Y$. To produce such informal illustrations, one often visualises X and Y as intervals on the coordinate axes, and subsets A, B as subintervals; this motivates the terminology.

Note that not all subsets of $X \times Y$ are cylinder or rectangle sets.

Example: intersections of rectangle sets.

Show that the intersection of any collection of rectangle sets is a rectangle set. Show that a union of rectangle sets may not be a rectangle set.

Solution: we calculate the intersection of two rectangle sets $A \times B$ and $A' \times B'$ where $A, A' \subseteq X$ and $B, B' \subseteq Y$. We have

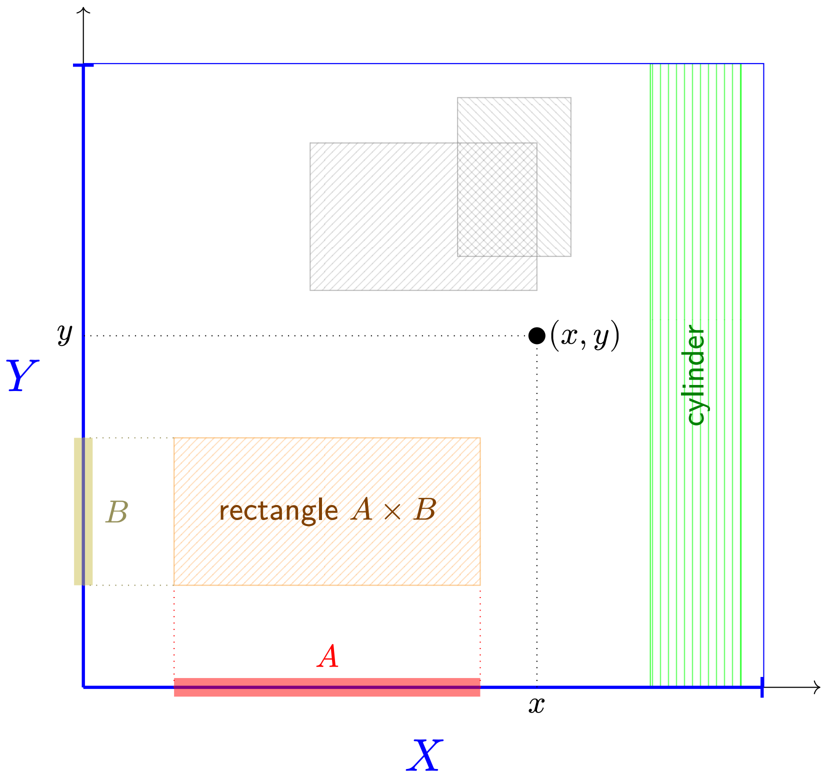

Figure 8.1: *rectangle and cylinder sets in $X \times Y$. The union of rectangles may not be a rectangle, but the intersection always is.*

$$
\begin{aligned}
&(A \times B) \cap (A' \times B') = \{(x, y) : (x \in A \text{ and } y \in B) \text{ and } (x \in A' \text{ and } y \in B')\} \\
&= \{(x, y) : x \in A \text{ and } x \in A' \text{ and } y \in B \text{ and } y \in B'\} \\
&= (A \cap A') \times (B \cap B'), \text{ a rectangle set.}
\end{aligned}
$$

In the same way one shows, for any collection $\{A_\alpha \times B_\alpha\}_{\alpha \in I}$ of rectangle sets, that

$$\bigcap_{\alpha \in I} (A_\alpha \times B_\alpha) = (\bigcap_{\alpha \in I} A_\alpha) \times (\bigcap_{\alpha \in I} B_\alpha),$$

i.e., the intersection is a rectangle set. Yet Figure 8.1 shows an example of two rectangle sets (with grey pattern) whose **union** is **not** a rectangle set.

The product topology

From now on, we assume that X and Y are not just sets but **topological spaces.** Consider the collection

$$\mathscr{B} = \{U \times V : U \subseteq X \text{ is open in } X,\ V \subseteq Y \text{ is open in } Y\}$$

of subsets called **open rectangles** in $X \times Y$.

Definition: product topology on $X \times Y$.

The **product topology** on $X \times Y$ is the topology with base $\mathscr{B}$ of open rectangles. The set $X \times Y$ with this topology is the **product space** of X and Y.

Remark: one needs to show that $\mathscr{B}$ is indeed a base of some topology. This means checking that the intersection of two sets from $\mathscr{B}$ can be written as a union of sets from $\mathscr{B}$. But here, an even stronger statement holds: $(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V')$, that is, an intersection of two open rectangles is an open rectangle (no need to write it as a union of some collection of open rectangles).

We omit a full formal argument showing that $\mathscr{B}$ is a base of a topology; interested students can find it in the literature.

Alert.

Not all open sets in $X \times Y$ are open rectangles $U \times V$. Open sets are **arbitrary unions** of open rectangles.

The Euclidean plane is our first expected example of a product space.

Example.

Show that the metric Euclidean topology on $\mathbb{R}^2$ is the product topology on $\mathbb{R} \times \mathbb{R}$.

Solution (*not given in class*)**:** denote the product topology by $\mathscr{T}_{\mathbb{R}\times\mathbb{R}}$ and the metric Euclidean topology by $\mathscr{T}_{\text{metric}}$. Every open rectangle from the base $\mathscr{B}$ of $\mathscr{T}_{\mathbb{R}\times\mathbb{R}}$ is open in $\mathscr{T}_{\text{metric}}$, hence $\mathscr{T}_{\text{metric}}$ is stronger than $\mathscr{T}_{\mathbb{R}\times\mathbb{R}}$. On the other hand, we saw earlier that $\mathscr{T}_{\text{metric}}$ can be defined by the metric $d_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_2 - y_2|)$ and so has base of open squares $B_r((x, y)) = (x-r, x+r)\times(y-r, y+r)$ which is a subcollection of $\mathscr{B}$. Hence $\mathscr{T}_{\text{metric}}$ is weaker than $\mathscr{T}_{\mathbb{R}\times\mathbb{R}}$. We conclude that $\mathscr{T}_{\text{metric}} = \mathscr{T}_{\mathbb{R}\times\mathbb{R}}$. □

The product space $X \times Y$ comes equipped with two continuous maps.

> **Proposition 8.1: projections are continuous.**
>
> Given a product space $X \times Y$, the following **projection maps** are continuous:
>
> $p_X \colon X \times Y \to X$, $(x, y) \mapsto x$, and
>
> $p_Y \colon X \times Y \to Y$, $(x, y) \mapsto y$.

Proof. If $U \subseteq X$ is open, $p_X^{-1}(U) = U \times Y$. This is an open rectangle, hence an open set in $X \times Y$ by definition of the product topology. We have thus verified the definition of "p_X is continuous". The proof for p_Y is similar. □

References for the week 8 notes

The definition of **product topology** on $X \times Y$ and a formal proof that the collection $\mathscr{B}$ of open rectangles in $X \times Y$ is a base of a topology are given in [Sutherland, Proposition 10.9]. Our Proposition 8.1, **the projections are continuous**, is [Sutherland, Proposition 10.10].