Week 7

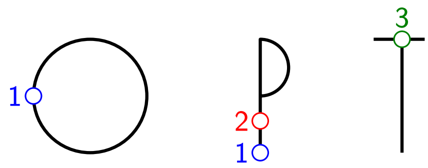
Exercises (answers at end)

Version 2024/11/20. To accessible online version of these exercises

Exercise 7.0. This is an unseen exercise in Applied Topology. In the diagram below, each letter of the English alphabet is drawn as a union of straight line segments and arcs.

Some letters are homeomorphic: for example, C $\cong$ J, both are homeomorphic to a closed interval. **Consider such homeomorphisms to be geometrically obvious.**

Some letters are **not** homeomorphic: here is a topological property that can distinguish them. If X is a topological space, call $p \in X$ **a point of connectivity k** if $X \setminus \{p\}$ has exactly k connected components. The following is easy to prove: *any homeomorphism $X \xrightarrow{\sim} Y$ maps a point of connectivity k to a point of connectivity k.* Hence, for each k, **the number of points of connectivity k is a topological property.** Example:



O $\not\cong$ P: O has no points of connectivity 2 but P has them;
T $\not\cong$ O and T $\not\cong$ P: T has a point of connectivity 3 while O, P have no such points.

CHALLENGE. Sort the letters into homeomorphism classes. You should have 9 classes.

Class 1:

Class 2:

Class 3:

Class 4:

Class 5:

Class 6:

Class 7:

Class 8:

Class 9:

Exercise 7.1. Consider the topological space $\mathbb{Q}$ which is the set of all rational numbers, viewed as a subspace of the Euclidean real line $\mathbb{R}$.

1. Is $\mathbb{Q}$ Hausdorff? Is $\mathbb{Q}$ compact? Justify your answer.
2. Show that the topology on $\mathbb{Q}$ is **not** discrete.
3. A topological space X is called **totally disconnected** if every non-empty connected subset of X is a singleton. Show that $\mathbb{Q}$ is totally disconnected.

Week 7

Exercises — solutions

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Some letters are homeomorphic: for example, C $\cong$ J, both are homeomorphic to a closed interval. **Consider such homeomorphisms to be geometrically obvious.**

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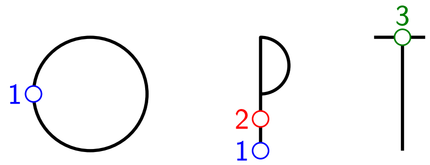

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CHALLENGE. Sort the letters into homeomorphism classes. You should have 9 classes.

Class 1: have a point of connectivity 4: K X

Class 2: has two points of connectivity 3: H

Class 3: one point of connectivity 3, three points of connectivity 1: E F T Y

Class 4: one point of connectivity 3, infinitely many points of connectivity 1: Q R

Class 5: one point of connectivity 2: B

Class 6: the set of points of connectivity 2 is disconnected: A

Class 7: the set of points of connectivity 2 is connected: P

Class 8: intervals — two points of connectivity 1, all other points are of connectivity 2: C G I J L M N S U V W Z

Class 9: circles — all points are of connectivity 1: D O

Exercise 7.1. Consider the topological space $\mathbb{Q}$ which is the set of all rational numbers, viewed as a subspace of the Euclidean real line $\mathbb{R}$.

1. Is $\mathbb{Q}$ Hausdorff? Is $\mathbb{Q}$ compact? Justify your answer.
2. Show that the topology on $\mathbb{Q}$ is **not** discrete.
3. A topological space X is called **totally disconnected** if every non-empty connected subset of X is a singleton. Show that $\mathbb{Q}$ is totally disconnected.

Answer to E7.1. 1. $\mathbb{Q}$ **is Hausdorff** because it is a subspace of a metric (hence Hausdorff) space $\mathbb{R}$. By Proposition 5.1, compacts in the metric space $\mathbb{R}$ must be closed and bounded. Since $\mathbb{Q}$ is not bounded, $\mathbb{Q}$ **is not compact.** (Another reason for non-compactness of $\mathbb{Q}$ is that $\mathbb{Q}$ is not closed in $\mathbb{R}$.)

2. Assume for contradiction that $\mathbb{Q}$ is discrete. Then $\{0\}$ must be an open subset of $\mathbb{Q}$, so by definition of subspace topology, $\{0\} = \mathbb{Q} \cap U$ where U is open in $\mathbb{R}$. By definition of

an open set in a metric space, U must contain the open ball $(-\varepsilon, \varepsilon)$ in $\mathbb{R}$ for some $\varepsilon > 0$. But any interval $(-\varepsilon, \varepsilon)$ of non-zero length contains infinitely many rational numbers, hence $\mathbb{Q} \cap U$ is infinite and not $\{0\}$. This contradiction shows that our assumption, "$\mathbb{Q}$ is discrete", was false.

3. Let $A \subseteq \mathbb{Q}$ be a non-empty connected set. The inclusion map $\mathrm{in}\colon \mathbb{Q} \to \mathbb{R}$ is continuous by Proposition 2.7, so $\mathrm{in}(A)$ is an interval in $\mathbb{R}$ by Proposition 5.3. Yet $\mathrm{in}(A) = A$, and non-empty intervals in $\mathbb{R}$ which consist entirely of rational points are singletons (every interval of non-zero length will contain irrationals). We have proved that A is a singleton.

References for the exercise sheet

E7.1 is an enhanced version of [Armstrong, Example 3 in Section 3.5].