Week 7

Connected components. Path-connectedness. Closure and interior

Version 2024/11/24 To accessible online version of this chapter

We continue to discuss connectedness.

Terminology.

We say **"A is a connected set in X"** or **"a connected subset of X"** to mean that A is a subset of a topological space X such that A, viewed with the subspace topology, is connected.

Theorem 7.1: a continuous image of a connected space is connected.

If X is a connected topological space and $f\colon X \to Y$ is continuous, then $f(X)$ is a connected set in Y.

Proof. Denote $Z = f(X)$. To prove that Z is connected using Proposition 5.3(ii), we need to assume that $h\colon Z \to \mathbb{R}$ is a continuous function, and to show that $h(Z)$ is an interval in $\mathbb{R}$. Considering the composite function $h \circ f\colon X \xrightarrow{f} f(X) = Z \xrightarrow{h} \mathbb{R}$ which is continuous by Proposition 2.6, one has $h(Z) = (h \circ f)(X)$. By Proposition 5.3(ii), $(h \circ f)(X) \subseteq \mathbb{R}$ is an interval. We have shown that $h(Z)$ is an interval, as required. $\square$

Remark (*not made in the lecture*): strictly speaking, in the proof we replaced the function $f\colon X \to Y$ by the function $f\colon X \to Z = f(X)$, which is known as **restricting the codomain.** We have to explain why the restricted-codomain function $X \xrightarrow{f} Z$ is still continuous. But this is easy: if $V \subseteq Z$ is a set **open in** Z, then V can be written as $Z \cap U$ where U is open in Y. One has $f^{-1}(V) = f^{-1}(Z \cap U)$. The preimage of the intersection is the intersection of preimages, so this equals $f^{-1}(Z) \cap f^{-1}(U) = X \cap f^{-1}(U) = f^{-1}(U)$ which is open in X as $X \xrightarrow{f} Y$ is given to be continuous. This shows that $X \xrightarrow{f} Z$ is continuous.

Corollary.

Connectedness is a topological property.

Proof. Replace the word "compact" with the word "connected" in the proof of the Corollary to Theorem 4.2. □

Connected components

A topological space may be disconnected, yet it is always made of connected "pieces" called connected components. To define these, we recall the notion of equivalence relation.

Notation.

A **relation** on a set X is any function $\sim\colon X \times X \to \{\text{True, False}\}$. We use infix notation for relations, writing "$\sim(x, y) = \text{True}$" as $x \sim y$ and "$\sim(x, y) = \text{False}$" as $x \nsim y$.

We have already verified the following definition for the relation "is homeomorphic to" on the class of all topological spaces. (Strictly speaking, this class is not a set, but we are going to ignore categorical subtleties here.) It is worth restating the definition more formally.

Definition: equivalence relation, equivalence class.

An **equivalence relation** relation on a set X is a relation $\sim$ such that $\sim$ is

- **reflexive:** $\forall x \in X$, $x \sim x$;
- **symmetric:** $\forall x, y \in X$, $x \sim y \Rightarrow y \sim x$;
- **transitive:** $\forall x, y, z \in X$, $(x \sim y) \wedge (y \sim z) \Rightarrow x \sim z$.

Suppose the above holds. For each $x \in X$, the subset

$$[x] = \{y \in X : x \sim y\}$$

of X is called the **equivalence class** of x.

We now introduce, on any topological space, an equivalence relation arising from connectedness.

Proposition 7.2: equivalence relation $\sim$ given by connectedness.

Let X be a topological space. For $x, y \in X$, let $x \sim y$ mean "there exists a connected set $A \subseteq X$ such that $x, y \in A$". Then $\sim$ is an equivalence relation on X.

Proof. **We prove that $\sim$ is reflexive:** let $x \in X$. Put $A = \{x\}$. Then A is a connected set: since A consists of only one point, A cannot be written as a union of two disjoint non-empty sets open in A. Since $x, x \in A$, we have $x \sim x$ by definition of $\sim$.

We prove that $\sim$ is symmetric: assume that $x, y \in X$ and $x \sim y$. Then there exists a connected set $A \subseteq X$ such that $x, y \in A$. The same can be written as $y, x \in A$, so $y \sim x$ by definition of $\sim$.

We prove that $\sim$ is transitive: assume that $x, y, z \in X$, $x \sim y$ and $y \sim z$. Then $x, y \in A$ and $y, z \in B$ where A and B are connected subsets of X. Note that $y \in A \cap B$ means that $A \cap B \neq \emptyset$, so by Lemma 7.3 below, the set $A \cup B$ is connected. Since $x, z \in A \cup B$, we have $x \sim z$ by definition of $\sim$. $\square$

Here is the lemma used in the proof of transitivity of $\sim$.

Lemma 7.3.

If A, B are connected subsets of X, $A \cap B \neq \emptyset$, then the union $A \cup B$ is connected.

More generally, if $\{A_\alpha : \alpha \in I\}$ is a collection of connected subsets of X such that $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$, then the union $\bigcup_{\alpha \in I} A_\alpha$ is connected.

Proof of the Lemma. Pick a point $y \in A \cap B$. We will use Proposition 5.3(iii) to show that $A \cup B$ is connected.

Let $g\colon A \cup B \to \{0, 1\}$ be any continuous function from $A \cup B$ to the discrete two-point space. The restriction $g|_A\colon A \to \{0, 1\}$ is a continuous function on A: indeed, $g|_A = g \circ \mathrm{in}_A$, the inclusion map in_A is continuous by Proposition 2.7, and the composition of continuous maps is continuous by Proposition 2.6. Since A is connected, by Proposition 5.3(iii) the function $g|_A$ is constant on A: all of its values on A are equal to $g(y)$, that is, $g(A) = \{g(y)\}$.

In the same way, $g(B) = \{g(y)\}$. But then $g(A \cup B) = g(A) \cup g(B) = \{g(y)\}$. We have proved that g is constant. This shows that $A \cup B$ is connected, by Proposition 5.3(iii).

The "more generally" part is proved similarly (*not in class*) and is left to the student. $\square$

The equivalence classes defined by $\sim$ have a special name:

Definition: connected components.

Let $x \sim y$ be the relation "$\exists A \subseteq X$: $x, y \in A$, A is connected" on a topological space X. The equivalence classes defined by $\sim$ are called **connected components** of X.

Recall that a **partition** of a set X is a collection of subsets of X which are non-empty, disjoint, and cover X. This is detailed in the following Claim, which is a well-known result from Mathematical Foundations.

Claim: equivalence classes form a partition.

If $\sim$ is an equivalence relation on a set X, the collection of equivalence classes $[x]$, where $x \in X$, forms a **partition** of the set X. That is,

- $[x]$ is non-empty for all x;
- either $[x] = [y]$ (equality of sets) or $[x] \cap [y] = \emptyset$, for all $x, y \in X$;
- $\bigcup_{x \in X} [x] = X$. □

Corollary.

Connected components of a topological space X form a partition of X. That is, X is a union of disjoint connected components.

The words "connected component" suggest that the set we are talking about is connected. This is indeed the case. *The following result was not proved in class.*

Lemma 7.4.

Each connected component of a topological space X is a connected subset of X.

Sketch of proof. The connected component $[x]$ of a point $x \in X$ is the union of all connected sets A in X such that $x \in A$. The intersection of all such sets contains x, hence their union is connected by the second statement of Lemma 7.3. □

Proposition 7.5: homeomorphism preserves connected components.

If $h\colon X \xrightarrow{\sim} Y$ is a homeomorphism, h maps connected components of X to connected components of Y.

Proof (not given in class). Let $x \in X$. We denote the connected component of x by $[x]$. Denote $y = h(x)$. Since h is continuous, by Theorem 7.1 $h([x])$ is a connected subset of Y; it contains y, and so $h([x]) \subseteq [y]$.

Now, considering the continuous function h^{-1}, the same argument shows that $h^{-1}([y]) \subseteq [x]$, therefore $[y] \subseteq h([x])$. The two inclusions mean that $h([x]) = [y]$, as claimed. □

Corollary.

The number of connected components (or the cardinality of the set of connected components) is a topological property.

Idea of proof (not given in class). The Proposition implies that a homeomorphism $h\colon X \xrightarrow{\sim} Y$ defines a map $\{\text{connected components of } X\} \to \{\text{connected components of } Y\}$.

It is easy to see that this map must be a bijection, because h is. Hence the set of connected components of X must be equipotent with the set of connected components of any space homeomorphic to X. □

Path-connectedness

We can see from Proposition 5.3 that connectedness of a topological space X can be characterised in terms of functions **from** X **to other spaces** such as $\mathbb{R}$ or $\{0,1\}$. We will now consider a different topological property, expressed in terms of functions **to** X**.**

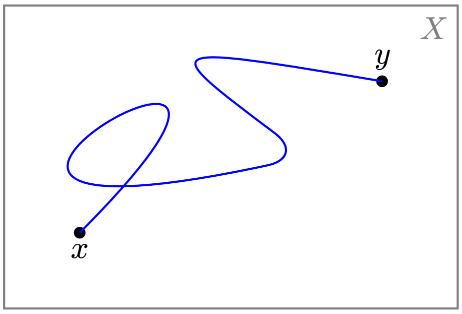
Definition: path; points joined by a path.

A **path** in a topological space X is a a continuous function $\phi\colon [0,1] \to X$. Points $x, y \in X$ are **joined by a path** if there exists a path ϕ with $\phi(0) = x$ and $\phi(1) = y$.

Here the closed interval $[0,1]$ is considered with the Euclidean topology. A path should be thought of as a continuous curve in X which starts at the point x and ends at the point y, see Figure 7.1 for an illustration.

Definition: a path-connected space.

A space X is **path-connected** if any two points of X are joined by a path.



[AI]

Figure 7.1: *"points x and y are joined by a path"*

Claim.

The continuous image of a path-connected space is path-connected. In particular, path-connectedness is a topological property.

Proof (not given in class). Suppose that X is a path-connected space and $f\colon X \to Y$ is continuous. To show that $Z = f(X)$ is path-connected, we pick $a, b \in Z$. We have $a = f(x)$ and $b = f(y)$ for some $x, y \in X$. Now let $\phi\colon [0,1] \to X$ be a path with $\phi(0) = x$ and $\phi(1) = y$.

The function $f \circ \phi$, where Z is taken as the codomain, is continuous, $(f \circ \phi)(0) = f(x) = a$ and $(f \circ \phi)(1) = f(y) = b$. Thus, $f \circ \phi$ is a path joining a and b in Z. □

Proposition 7.6: path-connected implies connected.

If a topological space X is path-connected, then X is connected.

Proof (not given in class). Assume X is path-connected, and fix a point $x \in X$. For any $y \in X$, let ϕ be a path joining x and y. Then x and y lie in the set $\phi([0,1])$ which is a connected set, being a continuous image of the connected interval $[0,1]$. Hence y lies in the connected component $[x]$ of x. Since y was arbitrary, this shows that X consists of only one connected component, and so X is connected by Lemma 7.4. □

Example.

Show that the Euclidean line $\mathbb{R}$ is not homeomorphic to the Euclidean plane $\mathbb{R}^2$.

Solution (*not given in class*)**:** assume for contradiction that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism. The set $\mathbb{R}^2 \setminus \{O\}$ is path-connected, see Figure 7.2: two points can be joined by a straight line segment or, if the segment contains O, by an arc; segments and arcs are paths. Since f is injective, we have $f(\mathbb{R}^2 \setminus \{O\}) = \mathbb{R} \setminus \{f(O)\}$. Yet by Proposition 5.3(ii), a continuous image of a connected space must be an interval in $\mathbb{R}$, which $\mathbb{R} \setminus \{\text{point}\}$ is not. This contradiction shows that a homeomorphism $\mathbb{R}^2 \to \mathbb{R}$ does not exist.

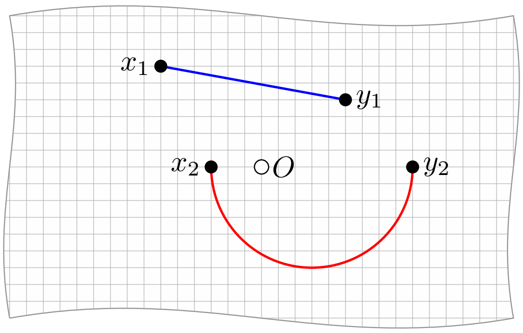


Figure 7.2: *the punctured plane* $\mathbb{R}^2 \setminus \{O\}$ *is path-connected*

Closure and interior

We now extend two constructions, introduced in MATH21111 *Metric Spaces*, to general topological spaces.

Definition: closure and interior of a set.

Let X be a topological space and A be a subset of X. The **closure** of A in X is

$$\overline{A} = \bigcap\{F : A \subseteq F,\ F \text{ is closed in } X\}.$$

The **interior** of A in X is

$$A^\circ = \bigcup\{U : U \subseteq A,\ U \text{ is open in } X\}.$$

In the next result, the **smallest** set in some collection of sets is the set (if it exists) which is contained in all other sets of the collection. Likewise, the **largest** set in a collection is the set which contains all other sets of the collection.

Claim 7.7.

$\overline{A}$ is the smallest closed subset of X which contains A.
A° is the largest open subset of X contained in A.

Proof. Let us denote by $\mathscr{C}_A$ the collection $\{F : A \subseteq F,\ F \text{ is closed in } X\}$. Then $\overline{A}$ is defined as $\bigcap \mathscr{C}_A$. We need to prove statements 1,2,3 as follows:

1. $\overline{A}$ **is closed in** X**.** Indeed, $\mathscr{C}_A$ is a collection of closed sets, hence by Proposition 2.4(b), the intersection $\overline{A}$ of $\mathscr{C}_A$ is closed.
2. $\overline{A}$ **contains** A**.** Indeed, each set in $\mathscr{C}_A$ contains A, and so $\bigcap \mathscr{C}_A$ also contains A.
3. $\overline{A} \subseteq G$ **for all** $G \in \mathscr{C}_A$**.** Indeed, $\overline{A} = \bigcap \mathscr{C}_A = G \cap \bigcap\{F \in \mathscr{C}_A : F \neq G\}$. Since $\overline{A}$ is the intersection of G with some set, we have $\overline{A} \subseteq G$, as claimed.

The claim about A° can be deduced from 1,2,3 above using the De Morgan laws 1.3: to do that, one shows that $A^\circ = X \setminus (\overline{X \setminus A})$. I leave this to the student. □

Corollary.

Let A be a subset of a topological space X. Then
(1) A is a closed set $\iff$ $A = \overline{A}$;
(2) A is an open set $\iff$ $A = A^\circ$.

Proof of Corollary. (1) $\Rightarrow$: assume A is closed in X. Then A is a closed set which contains A. By Claim 7.7, $\overline{A}$ is the smallest such set, so $\overline{A} \subseteq A$. On the other hand, also by Claim 7.7, $\overline{A} \supseteq A$. The two inclusions show that $\overline{A} = A$.

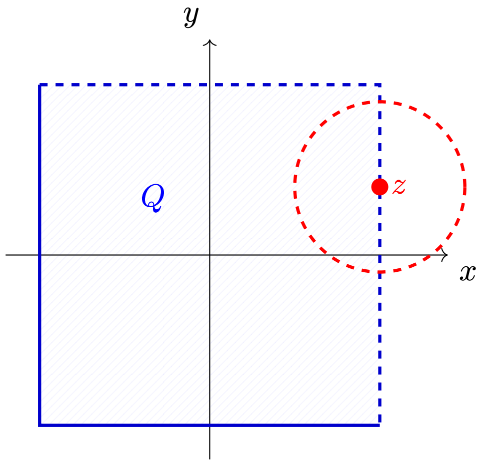


Figure 7.3: *z is a limit point for the half-open square $Q = \{-1 \le x < 1, \; -1 \le y < 1\}$; $z \notin Q$ but $z \in \overline{Q}$*

$\Leftarrow$: assume $A = \overline{A}$. By Claim 7.7, $\overline{A}$ is closed. Hence A is closed.

Part (2) is left to the student. $\square$

Closure as the set of "limit points"

We will now give another description of the closure of a set, based on the following:

Definition: limit point.

Let A be a subset of a topological space X. A point $z \in X$ is a **limit point** for A if $U \cap A \neq \emptyset$ for every open neighbourhood U of z.
In other words, **a point, whose every open neighbourhood meets A, is a limit point for A.**

It is obvious that if $z \in A$, then z is a limit point for A. The converse is false in general, see Figure 7.3 for illustration.

Proposition 7.8: closure equals the set of limit points.

$\overline{A} = \{z \in X : z \text{ is a limit point for } A\}$.

Proof. We will prove: $y \notin \overline{A} \iff y$ is not a limit point for A.

$\Rightarrow$: assume $y \notin \overline{A}$. Then y belongs to the set $U = X \setminus \overline{A}$. By Claim 7.7, U is open (as $\overline{A}$ is closed) and U does not meet A (as $A \subseteq \overline{A}$). Hence, by definition of a limit point, y is not a limit point for A, as claimed.

$\Leftarrow$: assume y is not a limit point for A, so that there is open $U \ni y$ with $U \cap A = \emptyset$. Then $X \setminus U$ is closed, and $A \subseteq X \setminus U$. By Claim 7.7, $\overline{A} \subseteq X \setminus U$, and since $y \in U$, we conclude that $y \notin \overline{A}$. □

We note that our definition of a limit point is not in terms of sequences. We will now define limits of sequences, in order to see the connection with Real Analysis and Metric Spaces.

Definition: convergence.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of a topological space X. We say that x_n **converges to a point** $x \in X$, and write $x_n \to x$ as $n \to \infty$, if for any open neighbourhood U of x there exists $N \in \mathbb{N}$ such that the tail $x_{N+1}, x_{N+2}, \ldots$ of the sequence (x_n) lies in U.

A sequence of points in a topological space may not converge to any point at all, converge to a single point, or converge to more than one point. This last option prevent us from saying "the limit of a sequence" because there might be more than one limit! This undesirable situation cannot occur in Hausdorff spaces:

Proposition 7.9: in Hausdorff, limit is unique if it exists.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Hausdorff space X, such that $x_n \to x$ and $x_n \to y$ as $n \to \infty$. Then $x = y$.

Proof (not given in class). Assume for contradiction that $x \neq y$. Since X is Hausdorff, $x \in U$ and $y \in V$ where U, V are disjoint open sets.

Since $x_n \to x$, there exists $M \in \mathbb{N}$ such that $x_M, x_{M+1}, \cdots \in U$. Likewise, there exists $N \in \mathbb{N}$ such that $x_N, x_{N+1}, \cdots \in V$. But then U and V are not disjoint, because both sets contain $x_{\max(M,N)+1}$. This contradiction shows that the assumption $x \neq y$ was false. □

Let A be a subset of a topological space X, and let $x \in X$. What is the relationship between the two statements,

(a) $x \in \overline{A}$;
(b) there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in A$ for all n, and $x_n \to x$ as $n \to \infty$.

In metric spaces, (a) and (b) are equivalent. In general topological spaces, (b) implies (a) but not the other way round. It turns out that the right condition for (a) and (b) to be equivalent is the following.

Definition: a first-countable space.

A topological space X is **first countable** if every point $x \in X$ has a countable system $U_1(x), U_2(x), \ldots$ of open neighbourhoods, such that the collection $\{U_n(x) : n \geq 1,\ x \in X\}$ is a base of topology on X.

All metric spaces are first countable: just put $U_n(x) = B_{\frac{1}{n}}(x)$.

We omit the proof of the following fact, which the students may wish to attempt as an exercise or look up in the literature.

Claim 7.10.

If X is a first-countable topological space and $A \subseteq X$, then $x \in \overline{A}$ iff there is a sequence $(x_n)_{n \in \mathbb{N}}$ contained in A which converges to x. (In particular, this is true for all metrisable topologies.)

The boundary of a set. Dense sets

We conclude the chapter with two definition which are important for normed, Hilbert and Banach spaces.

Definition: the boundary of a set.

Let X be a topological space and $A \subseteq X$. The **boundary of** A is the set $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

Combining this definition with Proposition 7.8, we arrive at the following equivalent description of the boundary of A:

> ∂A is the set of points $z \in X$ such that every open neighbourhood of z contains a point from A and a point not from A.

In Euclidean spaces, the notion of the boundary is quite intuitive. For example, the boundary of the half-open square $Q = \{(x, y) : -1 \le x < 1,\ -1 \le y < 1\}$ in the plane is exactly the "border", i.e., the union of the four sides, of the square: $\partial Q = \{(x, y) : \max(|x|, |y|) = 1\}$, see Figure 7.4 for illustration.

Definition: dense set.

Let X be a topological space. A subset A of X is **dense** in X if $\overline{A} = X$.

Of course, X is always dense in X. Yet smaller (e.g., countable) dense sets, if they exist, are usually more interesting. The following is a standard example from *Metric Spaces*:

Example: $\mathbb{Q}$ is dense in $\mathbb{R}$.

Show that the set $\mathbb{Q}$ of rational numbers is dense in the Euclidean line $\mathbb{R}$.

Solution (not given in class)**:** let $z \in \mathbb{R}$ be arbitrary. We need to show that $z \in \overline{\mathbb{Q}}$, which by Proposition 7.8 means that every open neighbourhood U of z meets $\mathbb{Q}$. Indeed, by definition of "open" in Euclidean topology, U contains an open interval $(z - \varepsilon, z + \varepsilon)$ for

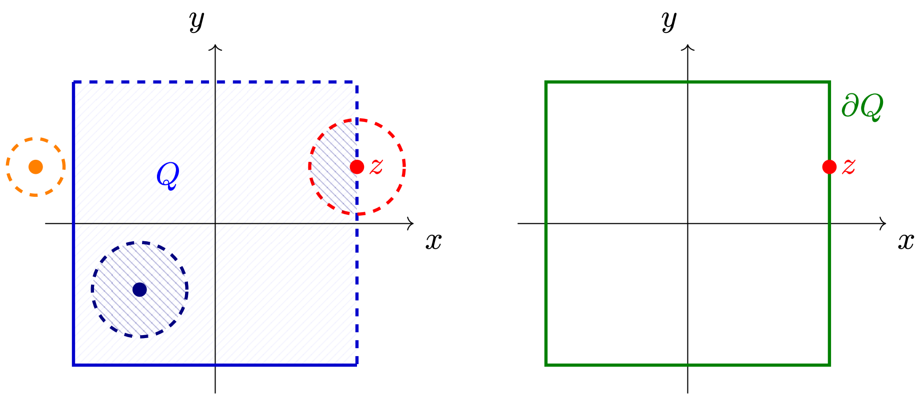


Figure 7.4: *the boundary of the half-open square* $Q = \{-1 \leq x < 1, -1 \leq y < 1\}$ *is the "border" of the square. Each open neighbourhood of a point on* ∂Q *intersects both* Q *and* $\mathbb{R}^2 \setminus Q$*. Non-boundary points have a neighbourhood fully in* Q *or fully in* $\mathbb{R}^2 \setminus Q$

some $\varepsilon > 0$, and it is a known fact that every interval of positive length in $\mathbb{R}$ contains rational points. □

The concepts of "connected" and "dense" lead to a well-known counterexample in topology, which we will now consider.

The rest of this chapter was not covered in class.

Lemma 7.11.

If a topological space X has a connected dense subset, then X is connected.

Proof. Let $A \subseteq X$ be such that $\overline{A} = X$. Assume that X is disconnected: that is, $X = U \cup V$ where U, V are disjoint non-empty sets open in X.

Take $x \in U$, so that U is an open neighborhood of x. Since $x \in X = \overline{A}$, by Proposition 7.8 we must have $U \cap A \neq \emptyset$. Taking $y \in V$, we similarly argue that $V \cap A \neq \emptyset$. Then $A = (U \cap A) \cup (V \cap A)$ is a disjoint union of non-empty sets, open in A; hence A is disconnected. The Lemma follows by contrapositive. □

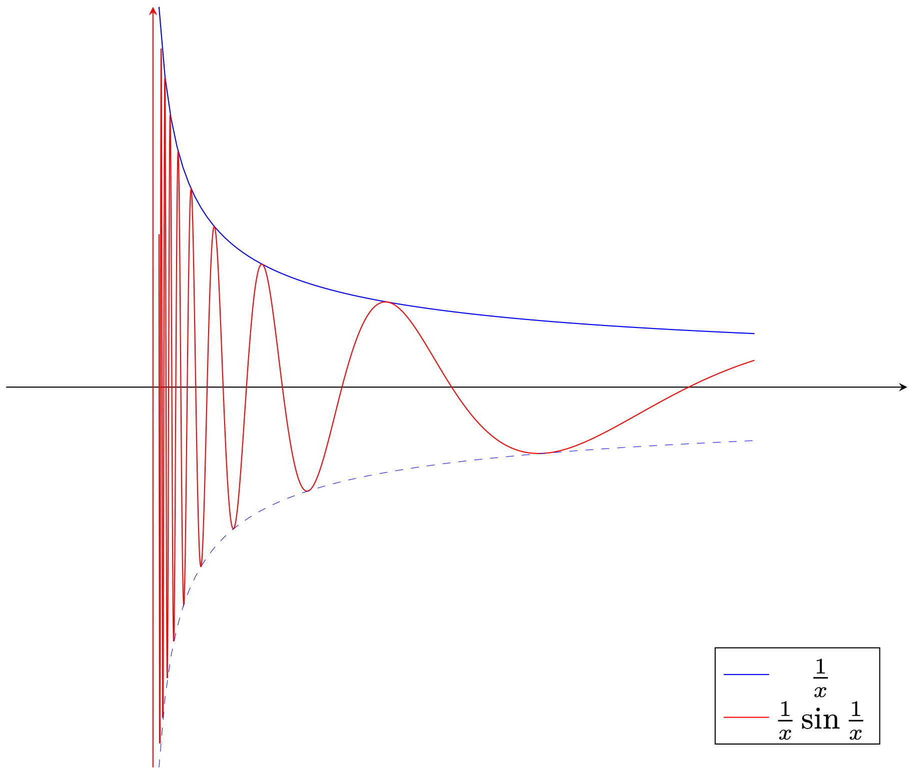


Figure 7.5: The (modified) topologist's sine curve

Example: Topologist's sine curve.

Let X be the subet $\{(x, y) : x = 0$ or $x > 0, y = \frac{1}{x}\sin\frac{1}{x}\}$ of the Euclidean plane $\mathbb{R}^2$, see Figure 7.5. Show that X is connected but not path-connected.

Solution. Let $X^+ = \{(x, y) : x > 0, y = \frac{1}{x}\sin\frac{1}{x}\}$ be the intersection of X with the positive half-plane $\{x > 0\}$. Then X^+ is the image of $(0, +\infty)$ under the continuous function $x \mapsto (x, \frac{1}{x}\sin\frac{1}{x})$ from $(0, +\infty)$ to $\mathbb{R}^2$. Since the interval $(0, +\infty)$ is connected, and a continuous image of a connected space is connected (Theorem 7.1), X^+ is connected.

It is clear that every point of the vertical axis $\{x = 0\}$ is a limit point of X^+, thus $X = \overline{X^+}$, and by Lemma 7.11, X is also connected.

Yet X is not path-connected. Indeed, assume for contradiction that there is a path

$\phi\colon [0,1] \to X$ such that $\phi(0) = (0,0)$ and $\phi(1) = (1, \sin 1)$; here $(1, \sin 1)$ is a point of X. Denote by p the projection $(x, y) \mapsto x$ which is continuous. Then $f = p \circ \phi$ is a continuous function $[0,1] \to [0, +\infty)$.

Since $[0,1]$ is connected, by Proposition 5.3(ii) $f([0,1])$ must be a real interval which contains $f(0) = 0$ and $f(1) = 1$. In particular, $f([0,1])$ contains $(0,1]$, which means that $\phi([0,1])$ contains a point of the form (t, y) for all $t \in (0,1]$. Such a point of X can only be $(t, \frac{1}{t} \sin \frac{1}{t})$. The y-coordinates of all such points are unbounded in $\mathbb{R}$, yet $\phi([0,1])$ must be compact by Theorem 4.2, hence bounded by Proposition 5.1. This contradiction shows that a path joining the points $(0,0)$ and $(1, \sin 1)$ inside X does not exist. $\square$

References for the week 7 notes

Theorem 7.1, **a continuous image of a connected space is connected**, is [Sutherland, Proposition 12.11], and the Corollary (**connectedness is a topological property**) is [Sutherland, Corollary 12.12].

Topology textbooks, such as [Sutherland] and [Armstrong], assume knowledge of equivalence relations. This topic is covered in introductory mathematics literature: for example, [Smith] defines an **equivalence relation** (Definition 1.6), **partition** (Def.1.9), **equivalence class** $[x]$ (Def.1.10), and proves our Claim that **equivalence classes form a partition** [Smith, Proposition 1.4].

Figure 7.1 is a TikZ diagram generated with the help of OpenAI ChatGPT.

Definitions of **two points joined by a path** and a **path-connected space** are [Sutherland, Definitions 12.20 and 12.21]. Proposition 7.6, **path-connected implies connected**, is [Sutherland, Proposition 12.23], but we give a shorter proof. The example showing that $\mathbb{R}$ is not homeomorphic to the $\mathbb{R}^2$ is given in the book before [Sutherland, Exercise 12.1].

A **limit point** is called “a point of closure” in [Sutherland, Definition 9.6], and $\overline{A}$ is **defined** as the set of points of closure for A. Under this approach, our Proposition 7.8 is just the definition, yet our definition of $\overline{A}$ as the intersection of a family of closed sets becomes a result which needs proof; see [Sutherland, Proposition 9.10].

Proposition 7.9, **in Hausdorff, limit is unique if it exists**, is [Sutherland, Proposition 11.4].

Theorem 2.31 in the 2023/24 notes for MATH21111 *Metric Spaces* says: y lies in $\overline{A}$ iff there exists a sequence $(y_n)_{n\geq 1}$ in A such that $y_n \to y$ as $n \to \infty$.

First-countable spaces are defined in [Willard, Definition 10.3]. Claim 7.10 is [Willard, Thm 10.4].

The topologist's sine curve is a well-known example of a connected space which is not path-connected. It is given in [Counterexamples in Topology, 118], although we slightly modify it multiplying $\sin \frac{1}{x}$ by $\frac{1}{x}$ to arrive at an easier contradiction via unboundedness. A similar example under the same name is [Willard, Example 27.3a].