Week 5

Exercises (answers at end)

Version 2024/10/30. To accessible online version of these exercises

Exercise 5.1. Let A be a subspace of a topological space X. Prove: if $F \subseteq A$ and F is closed in X, then F is closed in A.

Exercise 5.2 (unions and intersections of compact sets)**.** Let X be a topological space.

1. Show that a union of two compact subsets of X is compact.
2. Assuming that X is Hausdorff, show that an intersection of two compact subsets of X is compact. (Why do we need X to be Hausdorff?)

Exercise 5.3 (nested sequence of closed subsets in a compact)**.** Let K be a compact topological space. Assume that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$, where F_i is a non-empty closed subset of K for each $i \geq 1$. Prove that all the sets F_i have a common point.

Exercise 5.4 (a Hausdorff compact topology is "optimal")**.** Let $(X, \mathscr{T})$ be a Hausdorff compact topological space. Use the Topological Inverse Function Theorem to show that

1. any topology on X, which is strictly weaker than $\mathscr{T}$, is not Hausdorff;
2. any topology on X, which is strictly stronger than $\mathscr{T}$, is not compact.

Week 5

Exercises — solutions

Version 2024/10/30. To accessible online version of these exercises

Exercise 5.1. Let A be a subspace of a topological space X. Prove: if $F \subseteq A$ and F is closed in X, then F is closed in A.

Answer to E5.1. Assume that $F \subseteq A \subseteq X$ and that F is closed as a subset of X, meaning that $X \setminus F$ is open in X. To show that F is also closed as a subset of A, we write

$$A \setminus F = (X \setminus F) \cap A.$$

This means that by definition of subspace topology on A, the set $A \setminus F$ is open in A: this is a set of the form "(open in X) $\cap A$". Hence F is closed in A.

Exercise 5.2 (unions and intersections of compact sets)**.** Let X be a topological space.

1. Show that a union of two compact subsets of X is compact.
2. Assuming that X is Hausdorff, show that an intersection of two compact subsets of X is compact. (Why do we need X to be Hausdorff?)

Answer to E5.2. Let K and L be two compact subsets of X.

1. We use Criterion of Compactness for Subsets 4.1 to show that $M = K \cup L$ is compact. Suppose that M is covered by a collection $\mathscr{C}$ of sets open in X. Then $\mathscr{C}$ covers K (because $K \subseteq M$). Since K is compact, there exists a finite subcollection $U_1, \dots, U_m \in \mathscr{C}$ which still covers K.

In the same way, there exists a finite subcollection $V_1, \dots, V_n \in \mathscr{C}$ which covers L. Then the finite subcollection $U_1, \dots, U_m, V_1, \dots, V_n$ of $\mathscr{C}$ covers $K \cup L = M$. By constructing a finite subcollection of $\mathscr{C}$ which still covers M, we have verified the Criterion of Compactness for M; hence M is compact.

2. In a Hausdorff space, a compact set is closed, Proposition 4.4, hence both K and L are closed in X.

Intersections of closed sets are closed, Proposition 2.4, so $K \cap L$ is closed in X.

By the previous exercise, $K \cap L$ is also closed in K. A closed subset of a compact is compact, Proposition 4.3, so $K \cap L$ is compact.

Exercise 5.3 (nested sequence of closed subsets in a compact)**.** Let K be a compact topological space. Assume that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$, where F_i is a non-empty closed subset of K for each $i \geq 1$. Prove that all the sets F_i have a common point.

Answer to E5.3. Note that the question is about a collection of closed sets, whereas the definition of "compact" is in terms of open sets. The main idea is to **pass to the complement** and apply the De Morgan laws.

We are asked to prove that the intersection $\bigcap_{i=1}^{\infty} F_i$ is not empty. Equivalently, considering the complements $U_i = K \setminus F_i$ (which are **open**), we need to prove that **the union** $\bigcup_{i=1}^{\infty} U_i$ **is not the whole of K.**

Assume for contradiction that
$$\bigcup_{i=1}^{\infty} U_i = K.$$
Then the collection $U_1, U_2, \dots$ is an open cover of K. Since K is compact, there is a finite subcover $U_{i_1}, \dots, U_{i_n}$ so that
$$U_{i_1} \cup \dots \cup U_{i_n} = K.$$

Now note that $U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots$, and therefore $U_{i_1} \cup \cdots \cup U_{i_n} = U_j$ where $j = \max(i_1, \ldots, i_n)$. We have

$$U_j = K \quad \Rightarrow \quad F_j = \emptyset.$$

Yet the sets F_i were given to be non-empty. This contradiction shows that our **assumption** was false.

Exercise 5.4 (a Hausdorff compact topology is "optimal")**.** Let $(X, \mathscr{T})$ be a Hausdorff compact topological space. Use the Topological Inverse Function Theorem to show that

1. any topology on X, which is strictly weaker than $\mathscr{T}$, is not Hausdorff;
2. any topology on X, which is strictly stronger than $\mathscr{T}$, is not compact.

Answer to E5.4. 1. A topology $\mathscr{T}_w$ on X is weaker than $\mathscr{T}$ iff the identity function $\mathrm{id}_X \colon (X, \mathscr{T}) \to (X, \mathscr{T}_w)$ is continuous. (We note that the function id_X is always bijective.)

Assume that $(X, \mathscr{T})$ is compact and $(X, \mathscr{T}_w)$ is Hausdorff. Then by, $\mathscr{T}$IFT, the continuous bijection $\mathrm{id}_X \colon (X, \mathscr{T}) \to (X, \mathscr{T}_w)$ is a homeomorphism. That is, a set U is $\mathscr{T}$-open iff its image $\mathrm{id}_X(U) = U$ is $\mathscr{T}_w$-open. But this means that $\mathscr{T}_w = \mathscr{T}$. Hence it is not possible for a Hausdorff $\mathscr{T}_w$ to be strictly weaker than $\mathscr{T}$.

Part 2. is done in a similar way and is left to the student.

References for the exercise sheet

E5.1 is a variant of [Sutherland, Exercise 10.5]. E5.2 is [Sutherland, Exercises 13.3 and 13.10]. E5.3 is [Sutherland, Exercise 13.11].