Week 4

Exercises (answers at end)

Version 2024/11/26. To accessible online version of these exercises

Exercise 4.1 (basic test of openness)**.** Suppose that $\mathscr{B}$ is a base of a topology on X, and call the subsets of X which are members of $\mathscr{B}$ **basic open sets.**

Let A be a subset of X. Prove that the following are equivalent:

1. A is open in X.
2. A is a union of a collection of basic open sets.
3. For each point $x \in A$, there exists a basic open set U such that $x \in U$ and $U \subseteq A$.

Exercise 4.2 (the Euclidean topology has a countable base)**.** Consider the Euclidean space $\mathbb{R}^2$, and let $\mathscr{Q}$ be the (countable) collection of all open squares in $\mathbb{R}^2$ where the coordinates of all four vertices are rational numbers. Prove that $\mathscr{Q}$ is a base for the Euclidean topology.

Deduce that the collection of all open sets in the Euclidean space $\mathbb{R}^2$ has cardinality $\aleph$ (continuum), whereas the collection of all subsets of $\mathbb{R}^2$ has cardinality $2^{\aleph}$.

Reminder about cardinal numbers:

- $\aleph_0$ *(aleph-zero) denotes the countably infinite cardinality, e.g., the cardinality of* $\mathbb{N}$*;*

- $\aleph$ *(aleph) denotes the cardinality of continuum, e.g., the cardinality of* $\mathbb{R}$*,*
- *one has* $|\mathbb{R}| = \aleph = 2^{\aleph_0} = |P(\mathbb{N})| > \aleph_0$.

Exercise 4.3 (subbase)**.** Let $(Y, \mathscr{T})$ be a topological space. A **subbase** of $\mathscr{T}$ is a collection $\mathscr{S}$ of open sets such that **finite intersections of sets from $\mathscr{S}$ form a base of $\mathscr{T}$.**

It is worth noting that, given any set Y (without topology) and any collection $\mathscr{S}$ of subsets of Y, we can construct a topology $\mathscr{T}_{\mathscr{S}}$ on X by using $\mathscr{S}$ as a subbase. That is, $\mathscr{T}_{\mathscr{S}}$ consists of arbitrary unions of finite intersections of members of $\mathscr{S}$. It is not difficult to show that this collection $\mathscr{T}_{\mathscr{S}}$ is a topology.

Prove that the collection of all **open rays** in the real line, i.e., sets of the form $(-\infty, a)$ and $(b, +\infty)$, is a subbase of the Euclidean topology.

Exercise 4.4 (subbasic test of continuity)**.** Let X, Y be topological spaces, $f\colon X \to Y$ be a function, and $\mathscr{S}$ be a subbase of topology on Y. Prove that the following are equivalent:

1. f is continuous.
2. The preimage of every subbasic set in Y is open in X (meaning: $\forall V \in \mathscr{S}$, $f^{-1}(V)$ is open in X.)

Exercise 4.5. (a) Let X be a topological space and let $f\colon X \to \mathbb{R}$ be a function. Prove: f is continuous iff for all $a, b \in \mathbb{R}$, the sets $X_{f<a} = \{x \in X : f(x) < a\}$ and $X_{f>b} = \{x \in X : f(x) > b\}$ are open in X.

(b) Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous functions. Prove that the function $f + g\colon X \to \mathbb{R}$ is continuous. Hint: use (a).

Week 4

Exercises — solutions

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Exercise 4.1 (basic test of openness)**.** Suppose that $\mathscr{B}$ is a base of a topology on X, and call the subsets of X which are members of $\mathscr{B}$ **basic open sets.**

Let A be a subset of X. Prove that the following are equivalent:

1. A is open in X.
2. A is a union of a collection of basic open sets.
3. For each point $x \in A$, there exists a basic open set U such that $x \in U$ and $U \subseteq A$.

Answer to E4.1. 1.$\Longleftrightarrow$2. by definition of a base.

Proof that 2.$\Rightarrow$3.: assume that $A = \bigcup \mathscr{C}$ where $\mathscr{C}$ is a collection of basic open sets. Let $x \in A$. By definition of union of a collection of sets, x is contained in at least one set in the collection $\mathscr{C}$; call this set U. The choice of U ensures that

- $x \in U$;
- $U \in \mathscr{C}$, and $\mathscr{C}$ is a collection of basic open sets, so U is a basic open set;
- $U \in \mathscr{C}$, and $\bigcup \mathscr{C} = A$, so $U \subseteq A$.

We have proved that 3. holds.

Proof that 3.⇒2.: assume that 3. holds. For each point $x \in A$, 3. allows us to choose a basic open set U_x such that $x \in U_x \subseteq A$.

We claim that the union of the collection $\{U_x\}_{x\in A}$ of basic open sets is A. Indeed,

- for all $y \in A$, we have $y \in U_y$ by the choice of U_y, hence $y \in \bigcup_{x\in A} U_x$; this proves that $A \subseteq \bigcup_{x\in A} U_x$;
- for each $x \in A$, $U_x \subseteq A$, and so $\bigcup_{x\in A} U_x \subseteq A$.

Thus $\bigcup_{x\in A} U_x = A$, and so 2. holds.

Exercise 4.2 (the Euclidean topology has a countable base)**.** Consider the Euclidean space $\mathbb{R}^2$, and let $\mathcal{Q}$ be the (countable) collection of all open squares in $\mathbb{R}^2$ where the coordinates of all four vertices are rational numbers. Prove that $\mathcal{Q}$ is a base for the Euclidean topology.

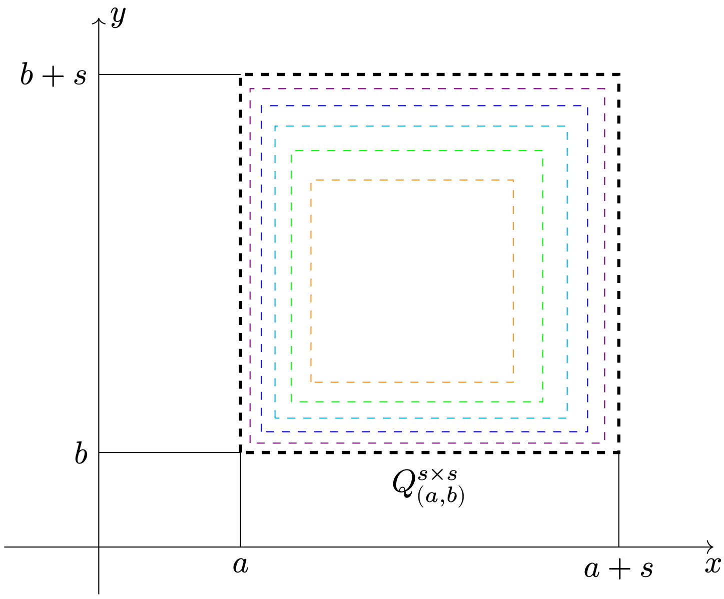
Deduce that the collection of all open sets in the Euclidean space $\mathbb{R}^2$ has cardinality $\aleph$ (continuum), whereas the collection of all subsets of $\mathbb{R}^2$ has cardinality $2^{\aleph}$.

Answer to E4.2. Denote by $Q^{s\times s}_{(a,b)}$ the square of size $s \times s$ whose bottom left corner is the point (a, b) in $\mathbb{R}^2$.

1. First, we show that every square $Q^{s\times s}_{(a,b)}$ with a, b, s real is a union of some collection $\{Q^{s_n\times s_n}_{(a_n,b_n)}\}_{n\geq 1}$ of squares with a_n, b_n, s_n rational.

Indeed, let (a_n) be a sequence of rational numbers such that $a_n \geq a$ and $\lim_{n\to\infty} a_n = a$. Also, let (s_n) be a sequence of rational numbers such that $\lim s_n = s$ and $a_n + s_n \leq a + s$. (It is not difficult to show that such sequences of rational numbers exist.)

Then $Q^{s_n\times s_n}_{(a_n,b_n)} \subseteq Q^{s\times s}_{(a,b)}$ for all n, and $\bigcup_{n\geq 1}\{Q^{s_n\times s_n}_{(a_n,b_n)}\} = Q^{s\times s}_{(a,b)}$. See the Figure for illustration.



Every square with sides parallel to the axes is a union of a collection of squares with rational coordinates

2. Now we argue that every set which is open in the Euclidean plane $\mathbb{R}^2$ is a union of some open squares. Recall that an open square plays the role of d_∞-open ball where the metric d_∞ on $\mathbb{R}^2$ is defined by

$$d_\infty((x_1, x_2), (y_1, y_2)) = \max(|x_1 - y_1|, |x_2 - y_2|),$$

see the discussion after Proposition 2.3. Specifically, $Q^{s\times s}_{(a,b)}$ is the open ball $B^{d_\infty}_r((a + \frac{s}{2}, b + \frac{s}{2}))$ where $r = \frac{s}{2}$.

By definition of metric topology, open balls form a base of topology so it follows that every d_∞-open set in $\mathbb{R}^2$ is a union of squares, and by 1., a union of rational squares.

It remains to recall that "d_∞-open" is the same as "Euclidean open", because the metric d_∞ is Lipschitz equivalent to the Euclidean metric d_2, see Proposition 2.3.

Exercise 4.3 (subbase)**.** Let $(Y, \mathscr{T})$ be a topological space. A **subbase** of $\mathscr{T}$ is a collection $\mathscr{S}$ of open sets such that **finite intersections of sets from $\mathscr{S}$ form a base of $\mathscr{T}$.**

It is worth noting that, given any set Y (without topology) and any collection $\mathscr{S}$ of subsets of Y, we can construct a topology $\mathscr{T}_{\mathscr{S}}$ on X by using $\mathscr{S}$ as a subbase. That is, $\mathscr{T}_{\mathscr{S}}$ consists of arbitrary unions of finite intersections of members of $\mathscr{S}$. It is not difficult to show that this collection $\mathscr{T}_{\mathscr{S}}$ is a topology.

Prove that the collection of all **open rays** in the real line, i.e., sets of the form $(-\infty, a)$ and $(b, +\infty)$, is a subbase of the Euclidean topology.

Answer to E4.3. Let $\mathscr{S}$ be the collection of all open rays in $\mathbb{R}$. By taking intersections of just two sets from $\mathscr{S}$, we can generate all open bounded intervals in $\mathbb{R}$:

$$(b, a) = (-\infty, a) \cap (b, +\infty).$$

Since the open intervals (b, a), where $a, b \in \mathbb{R}$, form a base of the Euclidean topology on $\mathbb{R}$, the topology $\mathscr{T}_{\mathscr{S}}$ generated by the subbase $\mathscr{S}$ contains the Euclidean topology.

On the other hand, every set in $\mathscr{S}$ is open in the Euclidean topology, hence so are unions of finite intersections of sets from $\mathscr{S}$. Therefore, the topology $\mathscr{T}_{\mathscr{S}}$ is contained in the Euclidean topology.

We conclude that $\mathscr{T}_{\mathscr{S}}$ is equal to the Euclidean topology, as claimed.

Exercise 4.4 (subbasic test of continuity)**.** Let X, Y be topological spaces, $f\colon X \to Y$ be a function, and $\mathscr{S}$ be a subbase of topology on Y. Prove that the following are equivalent:

1. f is continuous.
2. The preimage of every subbasic set in Y is open in X (meaning: $\forall V \in \mathscr{S}$, $f^{-1}(V)$ is open in X.)

Answer to E4.4. 1. $\Rightarrow$ 2.: by definition of subbase, $\mathscr{S}$ is a subcollection of the topology on Y, i.e., every subbasic set in is open in Y. By definition of "continuous", the preimage of an open set is open, and so the preimages of subbasic sets must be open in X, proving 2.

2. $\Rightarrow$ 1.: a base $\mathscr{B}$ of topology on Y consists of sets of the form $V_1 \cap \dots \cap V_n$, where $n \geq 0$ and $V_1, \dots, V_n \in \mathscr{S}$. The preimage of intersection is the intersection of preimages, so we

have

$$f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n),$$

and, since $f^{-1}(V_i)$ is open in X by 2., and a finite intersection of open sets is open, we conclude that $f^{-1}(V)$ is open in X for all $V \in \mathscr{B}$.

Finally, every open set in Y is a union of sets from $\mathscr{B}$, and the preimage of a union is the union of preimages. We conclude that $f^{-1}(\text{open set in } Y)$ is open in X, hence, by definition of "continuous", f is continuous, proving 1.

Exercise 4.5. (a) Let X be a topological space and let $f\colon X \to \mathbb{R}$ be a function. Prove: f is continuous iff for all $a, b \in \mathbb{R}$, the sets $X_{f<a} = \{x \in X : f(x) < a\}$ and $X_{f>b} = \{x \in X : f(x) > b\}$ are open in X.

(b) Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous functions. Prove that the function $f + g\colon X \to \mathbb{R}$ is continuous. Hint: use (a).

Answer to E4.5. (a) Note that $X_{f<a} = f^{-1}((-\infty, a))$ and $X_{f>b} = f^{-1}((b, +\infty))$. The sets $(-\infty, a)$ and $(b, +\infty)$ form a subbase of the Euclidean topology on $\mathbb{R}$ (see an earlier exercise). Hence by the subbasic test of continuity (see the previous exercise), f is continuous iff all the sets $X_{f<a}$ and $X_{f>b}$ are open.

(b) We need to prove that the sets $X_{f+g<a}$, $X_{f+g>b}$ are open for all $a, b \in \mathbb{R}$. Note that

$$f(x) + g(x) < a \quad \iff \quad \exists t \in \mathbb{R} : \ f(x) < t, \ g(x) < a - t.$$

Indeed, $\Leftarrow$ is obvious, and to see $\Rightarrow$, take t to be any real number in the interval $(f(x), a - g(x))$. The above rewrites in terms of sets as

$$X_{f+g<a} = \bigcup_{t \in \mathbb{R}} X_{f<t} \cap X_{g<a-t}.$$

Since f, g are continuous, by (a) the sets $X_{f<t}$ and $X_{g<a-t}$ are open in X; the intersection of two open sets is open, and the union of any collection of open sets is open, which shows that $X_{f+g<a}$ is open.

It is shown in the same way that $X_{f+g>b}$ is an open subset of X, for all $b \in \mathbb{R}$. We now use (a) again to conclude that $f + g$ is a continuous function.