Week 4

Compactness

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Reminder: an **open cover** of a topological space X is a collection $\mathscr{C}$ of subsets of X where, for each $U \in \mathscr{C}$, U is an open subset of X, and $\bigcup \mathscr{C} = X$.

Definition: subcover of an open cover.

A **subcover** of an open cover $\mathscr{C}$ of X is a subcollection of $\mathscr{C}$ which is still an open cover of X.

The following is one of the key notions of the course.

Definition: compact.

A topological space X is **compact** if every open cover of X has a finite subcover.

Compactness is a very powerful property, but it may require an effort to show directly that X is compact, beyond simple examples. Let us start with a **non-example:**

Example: a non-compact topological space.

Show that the Euclidean line $\mathbb{R}$ is not compact.

Solution: consider the collection

$$\mathscr{C} = \{B_r(0) : r > 0\}$$

which consists of all open intervals $(-r, r)$ with r positive. These intervals are open, and their the union contains all points of $\mathbb{R}$; that is, $\mathscr{C}$ is an open cover of $\mathbb{R}$.

Yet $\mathscr{C}$ has no finite subcover: any finite subcollection $\{B_{r_1}(0), \dots, B_{r_n}(0)\}$ of $\mathscr{C}$ has union equal to $B_R(0)$ where $R = \max(r_1, \dots, r_n)$, and this is not the whole of $\mathbb{R}$.

Thus, there is an open cover of $\mathbb{R}$ which has no finite subcover, so $\mathbb{R}$ is not compact.

At the moment, we can only give a very easy example of a compact space:

> **Example: a finite space is compact.**
>
> Let X be a finite set. Show that any topology on X is compact.

Solution: exercise.

> **Terminology.**
>
> We say **"a compact"** to refer to a compact topological space.
>
> We say **"K is a compact set in X"** or **"a compact subset of X"** to mean that K is a subset of a topological space X such that K, viewed with the subspace topology, is compact.

We will often deal with compact sets contained inside some topological space, and the following technical lemma will simplify proofs.

> **Lemma 4.1: criterion of compactness for a subset.**
>
> Let K be a subset of a topological space X. The following are equivalent:
>
> 1. K is a compact subset of X.
> 2. Any collection $\mathscr{F}$ of open sets in X, which covers K (that is, $K \subseteq \bigcup \mathscr{F}$), has a finite subcollection which still covers K.

Proof (not given in class). 1. $\Rightarrow$ 2.: suppose the subspace topology on K is compact, and let $\mathscr{F}$ be a collection of **open subsets of** X such that $K \subseteq \bigcup \mathscr{F}$. The collection $\mathscr{F}_K = \{U \cap K : U \in \mathscr{F}\}$ of subsets of K is clearly an open cover of K. By assumption, K is compact so this open cover must have a finite subcover, say $\{U_1 \cap K, \dots, U_n \cap K\}$. Then $\{U_1, \dots, U_n\}$ is a finite subcollection of $\mathscr{F}$ which still covers K.

2. $\Rightarrow$ 1.: to show that K is compact, we let $\mathscr{C}$ be an open cover of K. By definition of subspace topology, $\mathscr{C}$ is of the form $\{U_\alpha \cap K : \alpha \in I\}$ where U_α are open in X. Clearly, for $\mathscr{C}$ to cover K, one must have $K \subseteq \bigcup_{\alpha \in I} U_\alpha$.

By condition 2., the collection $\{U_\alpha : \alpha \in I\}$ has a finite subcollection, say $U_{\alpha_1}, \dots, U_{\alpha_n}$, which still covers K. Hence $\mathscr{C}$ has finite subcover $U_{\alpha_1} \cap K, \dots, U_{\alpha_n} \cap K$, verifying the definition of "compact" for K. □

The next result shows that compactness is not only a topological property but can help solve Main Problem 2, mentioned earlier.

Theorem 4.2: a continuous image of a compact is compact.

If X is a compact topological space and $f\colon X \to Y$ is continuous, then $f(X)$ is a compact set in Y.

Proof. We will use Criterion 4.1 of compactness for a subset to show that $f(X)$ is a compact set. Suppose a collection $\mathscr{G}$ of open sets in Y covers $f(X)$: that is, $f(X) \subseteq \bigcup \mathscr{G}$.

Consider the collection $\mathscr{C} = \{f^{-1}(V) : V \in \mathscr{G}\}$ of subsets of X. We claim that $\mathscr{C}$ is an open cover of X. If V is open in Y, "f is continuous" means that $f^{-1}(V)$ is open in X, so all members of $\mathscr{C}$ are open in X. Also,

$$X = f^{-1}(f(X)) \subseteq f^{-1}\left(\bigcup \mathscr{G}\right) = \bigcup\{f^{-1}(V) : V \in \mathscr{G}\}$$

which shows that $\mathscr{C}$ covers X.

Since X is compact, $\mathscr{C}$ has finite subcover, say $f^{-1}(V_1), \dots, f^{-1}(V_n)$. Then $V_1, \dots, V_n$ is a finite subcollection of $\mathscr{G}$ which covers $f(X)$. We have verified Criterion 4.1, hence $f(X)$ is a compact set in Y. □

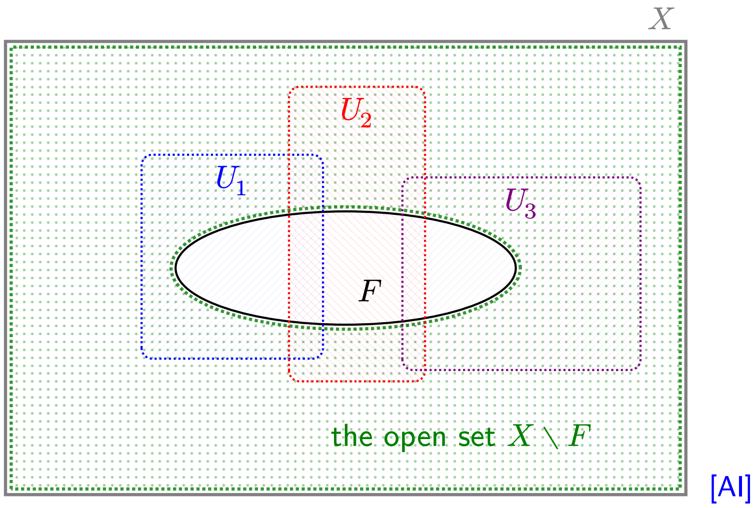

Figure 4.1: *if the sets $U_1, \dots, U_n, X \setminus F$ cover X, then the sets $U_1, \dots, U_n$ cover F*

Corollary.

Compactness is a topological property.

Proof. If X is compact and $X \underset{f}{\overset{\sim}{\to}} Y$, then f is continuous, so $f(X)$ must be compact by Theorem 4.2. Yet $f(X) = Y$ because f, being a homeomorphism, is surjective. □

If we found a compact space X, the next result allows us to construct new compact spaces.

Proposition 4.3.

A closed subset of a compact is compact.

Proof. Let X be a compact topological space and let F be a closed subset of X. We want to use Criterion 4.1, so we let F be covered by a family $\mathscr{C}$ of open subsets of X. Then

$$\mathscr{C} \cup \{X \setminus F\}$$

is an open cover for the whole of X.

Since X is compact, $\mathscr{C} \cup \{X \setminus F\}$ has a finite subcover of X. This finite subcover of X can be $U_1, \dots, U_n$ or $U_1, \dots, U_n, X \setminus F$. In either case, the sets $U_1, \dots, U_n$ form a finite subcollection of $\mathscr{C}$ which must cover F. (This last step is illustrated by Figure 4.1.) □

The above is as much as we can say about compact spaces without assuming additional topological properties besides compactness. We will now see that **compactness** works very well together with the **Hausdorff property:**

Proposition 4.4.

In a Hausdorff space, a compact set is closed.

Proof. Let X be a Hausdorff topological space and let K be a compact subset of X. Letting z be any point of $X \setminus K$, it is enough to prove:

$$(\dagger) \qquad z \in X \setminus K \quad \Rightarrow \quad \exists \text{ open } V(z)\text{: } z \in V(z) \text{ and } V(z) \subseteq X \setminus K.$$

Indeed, if (†) holds then, in the same way as in the proof of Proposition 3.4 $X \setminus K = \bigcup_{z \in X \setminus K} V(z)$ is an open set, so K is closed.

For each $x \in K$, the Hausdorff property gives us open neighbourhoods

$$U(x) \ni x, \quad V_x(z) \ni z : \quad U(x) \cap V_x(z) = \emptyset.$$

The open sets $\{U(x) : x \in K\}$ cover K, so by Criterion 4.1 there is a finite subcollection $U(x_1), \dots, U(x_n)$ which still covers K. Put

$$V(z) = V_{x_1}(z) \cap \dots \cap V_{x_n}(z).$$

(The construction of the open neighbourhood $V(z)$ is illustrated by Figure 4.2.)

As a finite intersection of open sets, $V(z)$ is open. Moreover, by construction $V(z) \cap U_{x_i}(z) \subseteq V_{x_i}(z) \cap U_{x_i}(z) = \emptyset$ and so

$$V(z) \cap (U(x_1) \cup \dots \cup U(x_n)) = \bigcup_{i=1}^{n} V(z) \cap U(x_i) = \emptyset.$$

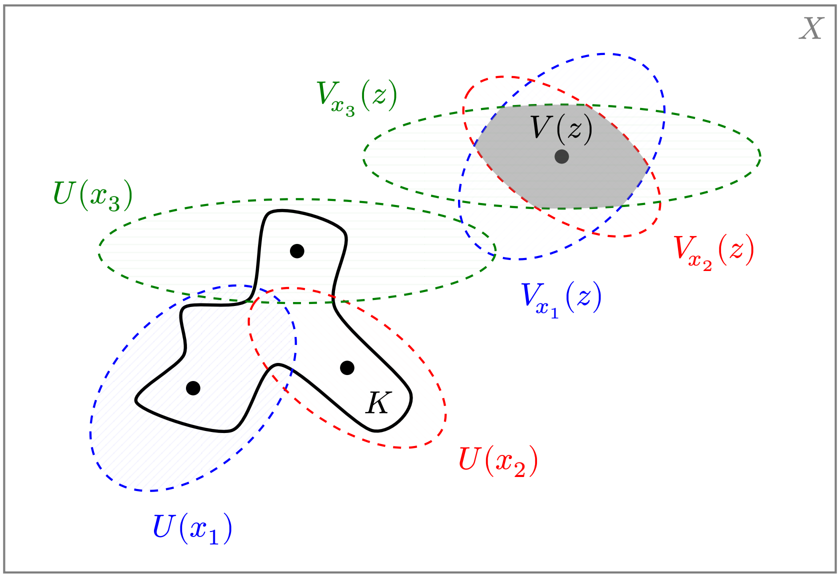

Figure 4.2: *the construction of the open neighbourhood $V(z)$ which does not intersect K*

Since K is contained in the union $U(x_1) \cup \cdots \cup U(x_n)$, it follows that $V(z)$ does not intersect K. We have therefore proved (†) and the Proposition. □

We arrive at a result which generalises important results from real analysis known as inverse function theorems.

Theorem 4.5: the Topological Inverse Function Theorem, $\mathscr{T}$IFT.

If K is a compact space, Y is a Hausdorff space and $f: K \to Y$ is a continuous bijection, then f is a homeomorphism.

Proof. f is already assumed to be bijective and continuous, hence to show that f is a homeomorphism, we need to prove that the inverse function $f^{-1}: Y \to K$ is continuous. We will use the closed set criterion of continuity, Proposition 2.5. Let $F \subset K$ be closed in K. The f^{-1}-preimage of F is $(f^{-1})^{-1}(F) = f(F)$:

- a closed subset of a compact is compact (Proposition 4.3) so F is compact,
- a continuous image of a compact is compact (Theorem 4.2), so $f(F)$ is compact,

- a compact subset of the Hausdorff space Y is closed (Proposition 4.4), so $f(F)$ is closed in Y.

We have shown that the function f^{-1} is such that the preimage of a closed set is closed. Hence, by the closed set criterion of continuity, f^{-1} is continuous. □

References for the week 4 notes

[Sutherland] gives detailed definitions of **cover, subcover** and **open cover** in [Sutherland, Definitions 13.3-13.5] and then defines a **compact subset** of X straight away in [Sutherland, Definition 13.6], without defining a compact space first. In this way, [Sutherland] avoids the Criterion of Compactness for a subset 4.1 altogether — the Criterion becomes the definition of compactness!

Our Theorem 4.2, **a continuous image of a compact is compact,** is [Sutherland, Proposition 13.15].

The key idea behind **Figure 4.1** is by OpenAI ChatGPT (prompt: generate a diagram to illustrate the proof that a closed subset of a compact is compact). YB changed the shapes of sets to make the diagram less cluttered.

Proposition 4.3, **a closed subset of a compact is compact** is [Sutherland, Proposition 13.20]. Proposition 4.4, **a compact is closed in Hausdorff,** is [Sutherland, Proposition 13.12]. Theorem 4.5, **the topological inverse function theorem,** is [Sutherland, Proposition 13.26].