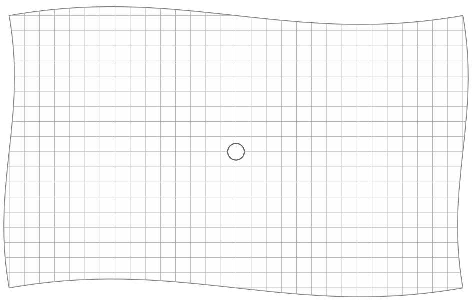
Week 3

Exercises (answers at end)

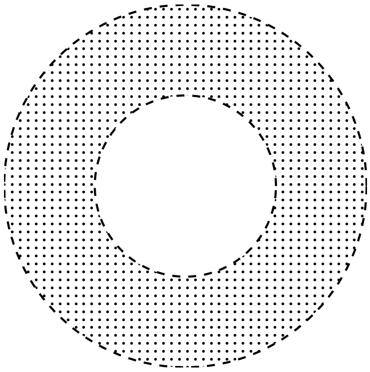
Version 2024/11/10. To accessible online version of these exercises

Exercise 3.1. Here is the unseen exercise done in the week 03 tutorial.

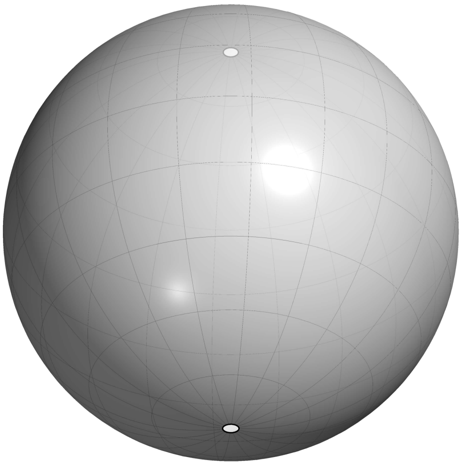
Consider the following topological spaces (the first five are viewed as subspaces of the Euclidean space). Determine which of these spaces are homeomorphic. Give a convincing description, or an explicit formula, for the homeomorphism where necessary; give reasons when the spaces are not homeomorphic.



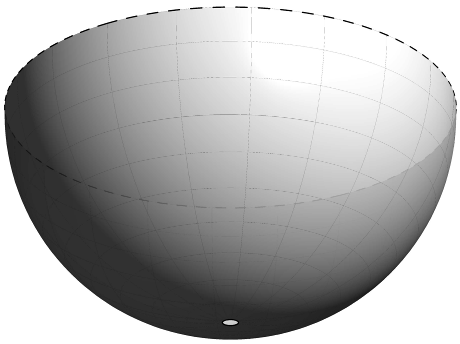
1. The punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$



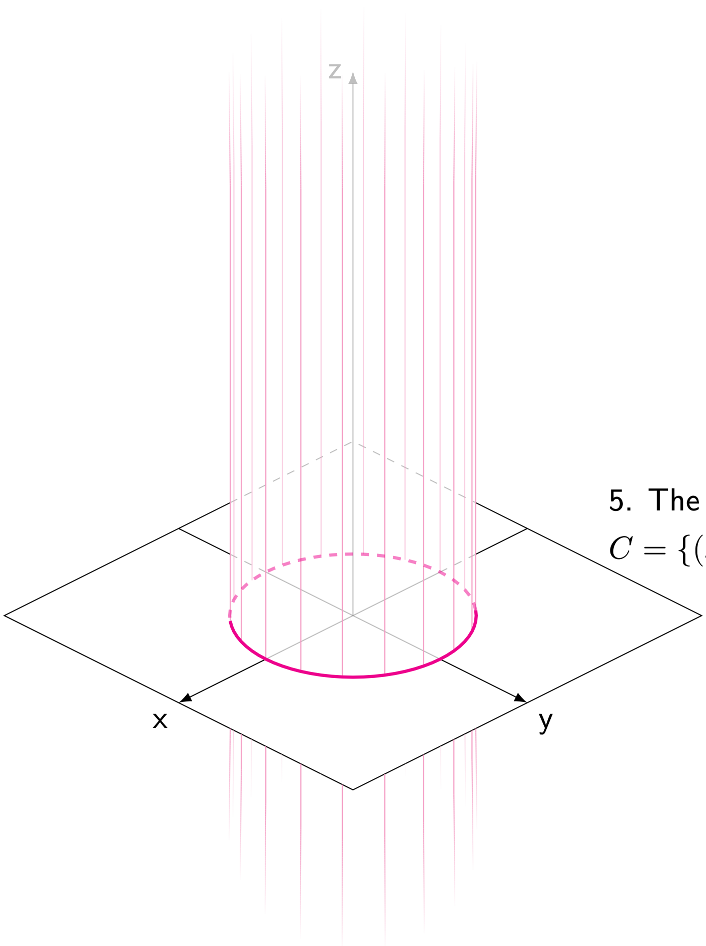
2. The open annulus
$A = \{(x, y) \in \mathbb{R}^2 : 1 < \sqrt{x^2 + y^2} < 2\}$



3. The twice-punctured sphere
$S^2 \setminus \{N, S\}$, where
$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$,
$N = (0, 0, 1)$ and $S = (0, 0, -1)$



4. The punctured open hemisphere
$(S^2 \cap \{z < 0\}) \setminus \{S\}$



5. The (infinite) cylinder
$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$

6. and finally, the set $\mathbb{R}^2$ with antidiscrete topology.

Week 3

Exercises — solutions

Version 2024/11/10. To accessible online version of these exercises

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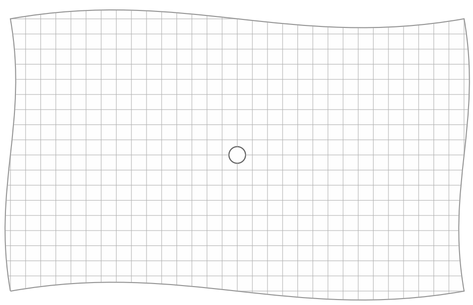

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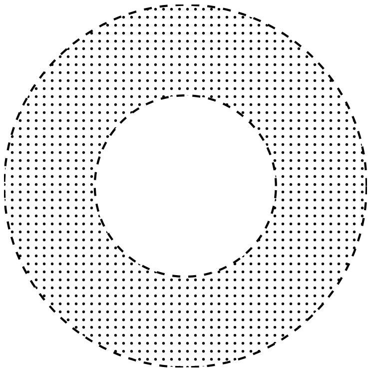

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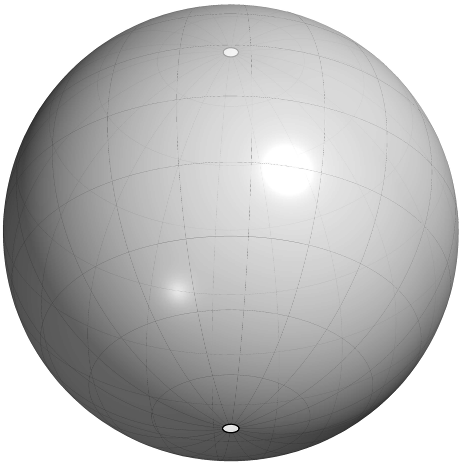

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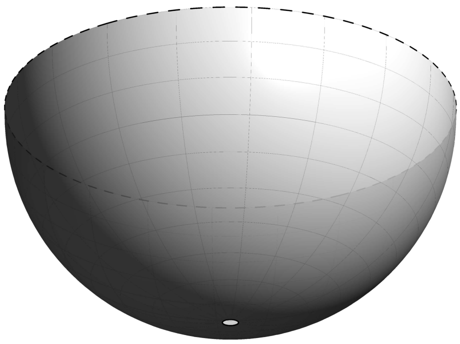

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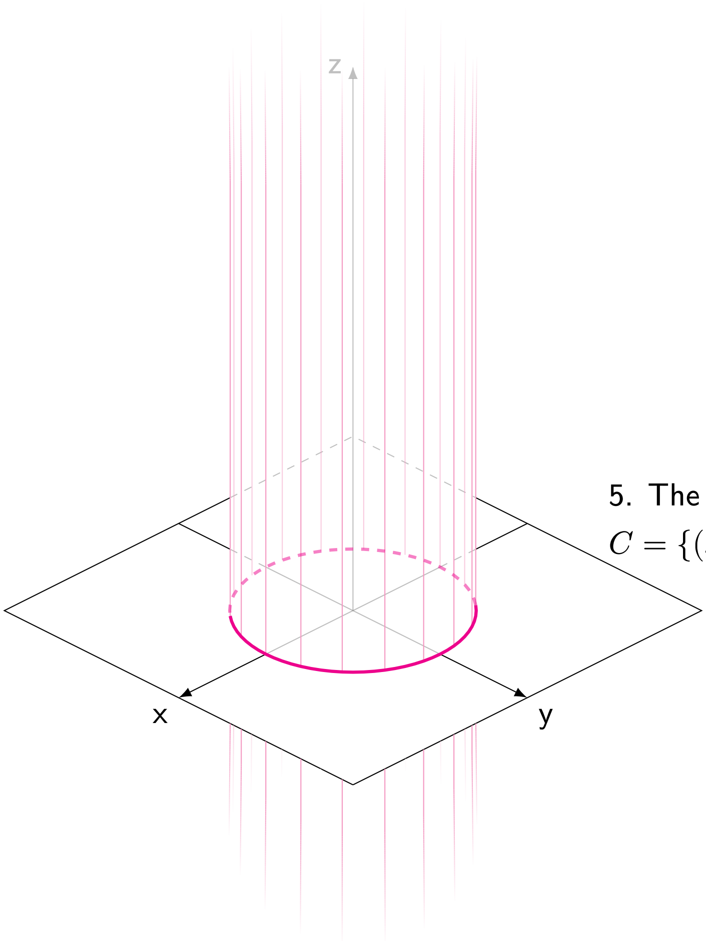

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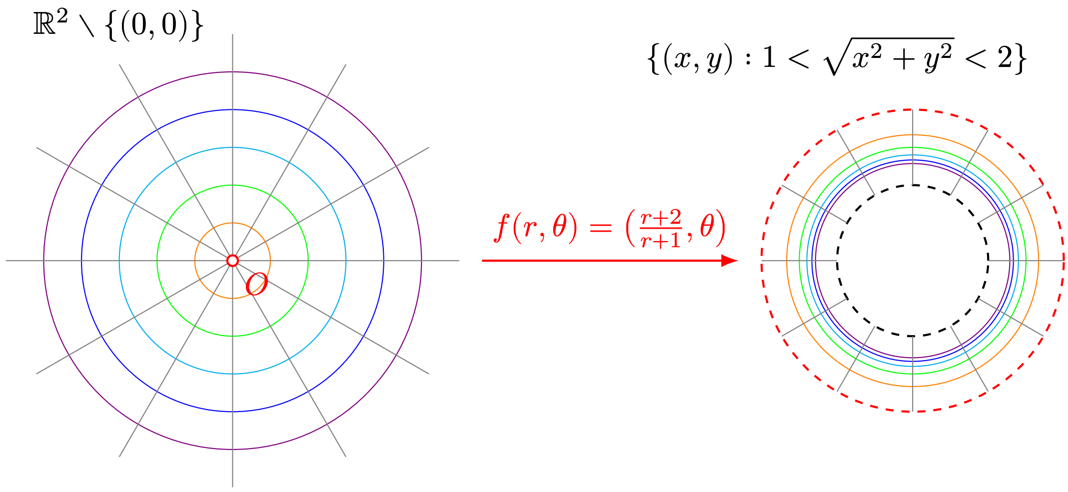
Answer to E3.1. **The first five topological spaces are homeomorphic:** we exhibit homeomorphisms between the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ and the other four spaces.

Punctured plane $\underset{f}{\overset{\sim}{\longrightarrow}}$ annulus: due to the rotational symmetry of both sets, it is convenient to define the continuous map f **in polar coordinates.** See the diagram in the Figure.

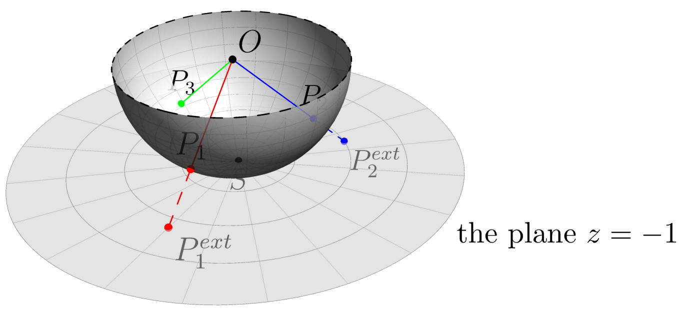
Twice-punctured sphere $\overset{\sim}{\longrightarrow}$ punctured plane: just use the stereographic projection, see Figure 3.2, which effects a homeomorphism between $\mathbb{R}^2$ and punctured sphere, and remains a homeomorphism if a point is removed from each space.



[AI]

Punctured hemisphere $\xrightarrow{\sim}$ punctured plane: modify the stereographic projection and project the hemisphere onto the plane from the origin O.

The diagram in the Figure shows how the homeomorphism is defined geometrically: if P is a point on the hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z < 0\}$, extend the line OP beyond P and let P^{ext} be the point of intersection of the extended line with the plane $\{z = -1\}$. The map $P \mapsto P^{ext}$ is the required homeomorphism, which remains such if the hemisphere and the plane are punctured by removing the point $S = (0, 0, -1)$.



[AI]

[Link to online interactive 3D diagram]

Punctured plane $\underset{g}{\xrightarrow{\sim}}$ cylinder: we again use polar coordinates (r, θ) on the plane. We use **cylindrical** coordinates (r, θ, z) in $\mathbb{R}^3$, where the equation of the cylinder is $r = 1$.

Informally, the punctured plane consists of open half-lines extending radially from the origin. We would like to map each such half-line, which is isomorphic to $(0, +\infty)$, onto a straight line on the side of the cylinder (a *generatrix* of the cylinder).

A possible homeomorphism between $(0, +\infty)$ and the Euclidean line $\mathbb{R}$ is given by the mutually inverse functions $\ln$ and $\exp$:

$$(0, +\infty) \xrightarrow{\ln} \mathbb{R} \xrightarrow{\exp} (0, +\infty).$$

This results in the following homeomorphism $g\colon \mathbb{R}^2 \setminus \{(0,0)\} \to$ cylinder, polar to cylindrical coordinates:

$$g\colon (r, \theta) \mapsto (1, \theta, \ln(r)).$$

The inverse map is $g^{-1}\colon (1, \theta, z) \mapsto (e^z, \theta)$, and both g and g^{-1} are clearly continuous because $\ln$ and $\exp$ are continuous. Hence g is a homeomorphism.

Since "homeomorphic" is an equivalence relation and in particular is transitive, we have done enough to show that the first five of the given spaces are pairwise homeomorphic. They are not homeomorphic to the remaining space, $\mathbb{R}^2$ with the antidiscrete topology. For example, they have the topological property that there exists an open set which is neither $\emptyset$ nor the whole space; $(\mathbb{R}^2$, antidiscrete$)$ does not have this property.

References for the exercise sheet

The homeomorphism $f(r, \theta)$ between the punctured plane and the annulus was worked out by OpenAI ChatGPT by improving on two incorrect attempts. It is instructive to read the full conversation with the AI chatbot. The diagram is based on AI-generated code but enhanced visually by YB.

A full proof that f is a homeomorphism can be read here. It is interesting to note that the AI-suggested formula, $r \mapsto \dfrac{r+2}{r+1}$, for the radial component of the map f is a decreasing function. Thus, points of the punctured plane close to the (cut-out) origin are sent by f to points on the annulus close to the outer boundary. Points of the punctured plane that are far away from the origin are sent by f to points close to the inner boundary of the annulus.

The homeomorphism between the punctured lower hemisphere and the punctured plane: to produce the 3D diagram, the open hemisphere drawing by YB was used by OpenAI ChatGPT to add the tangent plane and the visual representation of sample points and their projections.