Week 3

Homeomorphic spaces. Topological properties. Hausdorff spaces

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Homeomorphisms and homeomorphic spaces

We have arrived at one of the most important definitions of the course.

> **Definition: homeomorphism, homeomorphic spaces.**
>
> Let X and Y be topological spaces. A function $f\colon X \to Y$ is a **homeomorphism** if
>
> - f is a bijection,
> - f is continuous,
> - the inverse, $f^{-1}\colon Y \to X$, of f, is continuous.
>
> Two topological spaces X, Y are **homeomorphic** if there exists a homeomorphism $f\colon X \to Y$.

Note that the topologies of two homeomorphic spaces are considered equivalent. Indeed, if f is a homeomorphism between the topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$, then there is a one-to-one correspondence

$$\mathscr{T}_X \to \mathscr{T}_Y, \quad U \mapsto f(U)$$

between the two topologies. Indeed, if $U \subseteq X$ is open then $f(U) = (f^{-1})^{-1}(U)$ is also open, because f^{-1} is continuous by definition of a homeomorphism; and every open set $V \subseteq Y$ is an image, under f, of exactly one open set, $f^{-1}(V)$, in X.

> **Notation: homeomorphism.**
>
> We will write $X \xrightarrow{\sim} Y$ to indicate that there is a homeomorphism between X and Y, or $X \underset{f}{\xrightarrow{\sim}} Y$ to indicate that $f\colon X \to Y$ is a homeomorphism.

Earlier, we used the word "equivalent" to describe a relation between topological spaces which are homeomorphic. This word has a precise mathematical meaning:

> **Claim: 'homeomorphic' is an equivalence relation.**
>
> 'Homeomorphic' is an equivalence relation between topological spaces.

Proof. We need to check the three conditions of equivalence relation.

Reflexive: we need to show that every space X is homeomorphic to itself. Indeed, $X \underset{\mathrm{id}_X}{\xrightarrow{\sim}} X$. The identity map id_X is a bijection, is continuous as shown earlier, and $\mathrm{id}_X^{-1} = \mathrm{id}_X$ hence the inverse is also continuous.

Symmetric: We need to show that if $X \xrightarrow{\sim} Y$, then $Y \xrightarrow{\sim} X$. Assume that $X \underset{f}{\xrightarrow{\sim}} Y$. Since f is a homeomorphism, the inverse bijection f^{-1} is, by definition, continuous. Furthermore, the inverse of f^{-1} is f which is continuous. We conclude that $Y \underset{f^{-1}}{\xrightarrow{\sim}} X$.

Transitive: we need to show that if $X \xrightarrow{\sim} Y$ and $Y \xrightarrow{\sim} Z$, then $X \xrightarrow{\sim} Z$. Assume that $X \underset{f}{\xrightarrow{\sim}} Y \underset{g}{\xrightarrow{\sim}} Z$. Then $X \underset{g\circ f}{\xrightarrow{\sim}} Z$ where $g \circ f$ is a homeomorphism: it is a bijection as it has inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, and both $g \circ f$ and $f^{-1} \circ g^{-1}$ are continuous as compositions of continuous maps, by Proposition 2.6. □

Let us now consider examples of homeomorphic spaces.

> **Example: some spaces homeomorphic to $\mathbb{R}$.**
>
> Show that the Euclidean line $\mathbb{R}$ is homeomorphic to
> - the open interval $(-\pi/2, \pi/2)$ (a subspace of $\mathbb{R}$);
> - an open half-line $(0, +\infty)$;
> - the right open half-circle of the unit circle in the complex plane.

Solution. We construct pairs of continuous functions which are mutual inverses between some pairs of the given spaces:

$$\begin{array}{ccc} (-\pi/2, \pi/2) & \underset{\text{arctan}}{\overset{\tan}{\rightleftarrows}} & \mathbb{R} \\ {\scriptstyle \arg}\uparrow\downarrow{\scriptstyle \theta\mapsto e^{i\theta}} & & {\scriptstyle \ln}\uparrow\downarrow{\scriptstyle \exp} \\ \text{half-circle} & & (0, +\infty) \end{array}$$

We do not need to construct homeomorphisms between **each** pair of the given spaces: since "homeomorphic" is an equivalence relation, we have constructed enough to show that all four spaces are homeomorphic to each other.

The homeomorphism between the right half-circle of the unit circle and $\mathbb{R}$ can be described in purely geometric terms as in Figure 3.1: take a point of the half-circle and project it onto the vertical tangent line to the half-circle at the point $(1, 0)$ in the plane.

The homeomorphisms shown above are not unique, and there are many other ways of showing that these four spaces are pairwise homeomorphic.

Remark. The four spaces shown above are quite different as **metric spaces:** for example,

- $\mathbb{R}$ and $(0, +\infty)$ are unbounded but $(-\pi/2, \pi/2)$ and the half-circle are bounded;
- $\mathbb{R}$ is a complete metric space whereas $(0, +\infty)$, $(-\pi/2, \pi/2)$ and the open half-circle are not complete.

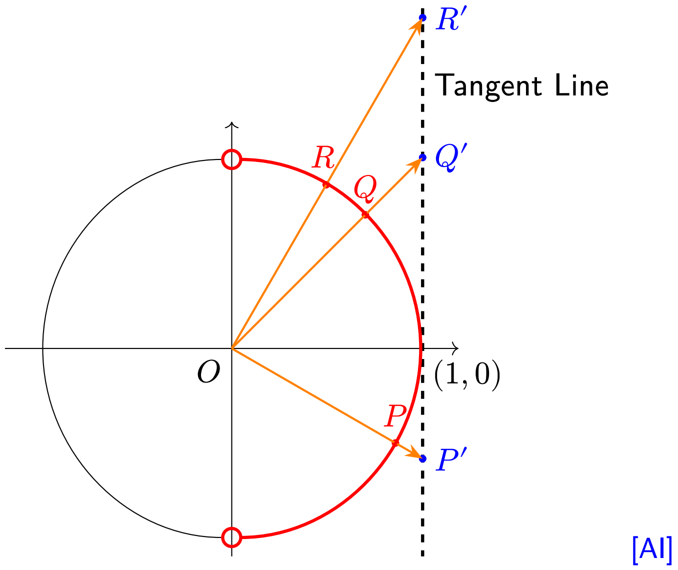


Figure 3.1: *a homeomorphism between the open half-circle and a straight line*

Hence there is **no way** these metric spaces can be **isometric** (i.e., equivalent as metric spaces). We thus observe that being **homeomorphic** (i.e., equivalent as topological spaces) is a weaker requirement, for metric spaces, than being isometric.

Example: some spaces homeomorphic to $\mathbb{R}^2$.

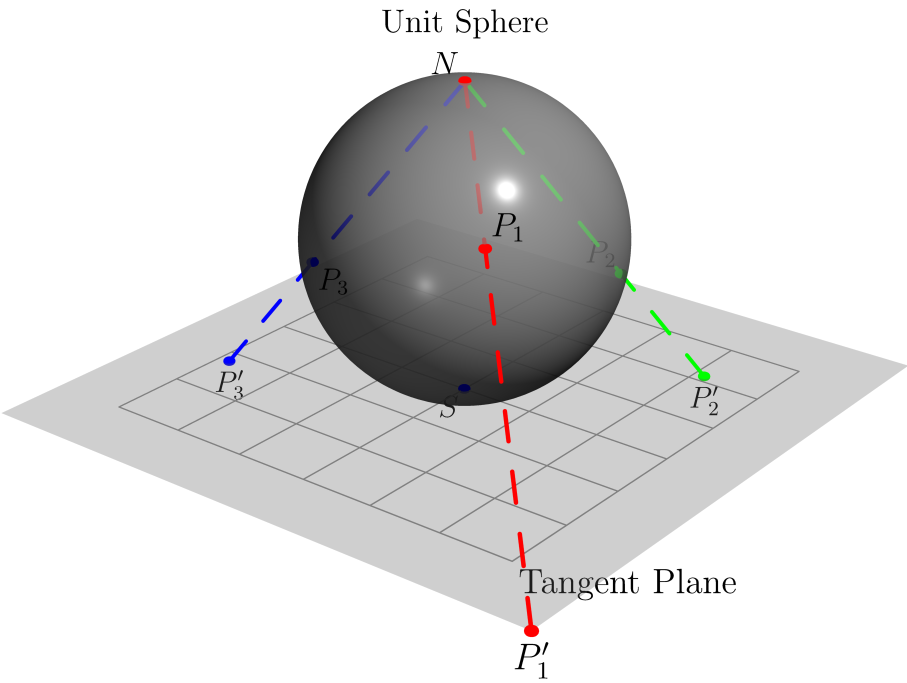
Show that the Euclidean plane $\mathbb{R}^2$ is homeomorphic to

- the **punctured sphere** (a sphere in $\mathbb{R}^3$ with one point removed);
- the open unit disc $B_1((0,0))$ in $\mathbb{R}^2$;
- the open quadrant $\{(x,y) \in \mathbb{R}^2 : x, y > 0\}$.

Solution: we exhibit a homeomorphism between $\mathbb{R}^2$ and a punctured sphere, leaving the other spaces to the student.

Consider the unit sphere $S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$, let N be the point $(0,0,1)$ — "the North Pole" of the sphere — and let S be the point $(0,0,-1)$, "the South pole". We construct

$$f\colon S^2 \setminus \{N\} \to \mathbb{R}^2,$$



[AI]

Figure 3.2: *the stereographic projection is a homeomorphism between a punctured sphere and* $\mathbb{R}^2$. [Link to online interactive 3D diagram]

where $\mathbb{R}^2$ is identified with the plane $\{z = -1\}$ tangent to the sphere at S. The function f is defined as follows:

- let P be a point on $S^2 \setminus \{N\}$;
- extend the straight line NP beyond P;
- let P' be the point of intersection of the line NP with the plane $\{z = -1\}$;
- put $f(P) = P'$.

The construction of f is illustrated by the interactive Figure 3.2. The map f is known as the **stereographic projection.** Homeomorphisms between subsets of the sphere and subsets of a plane have practical applications in **cartography** (the science of drawing maps).

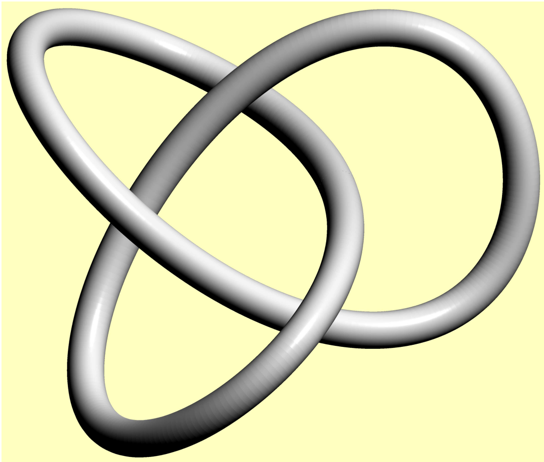


Figure 3.3: *a trefoil knot* [Link to online interactive 3D diagram]

Further examples and applications of homeomorphisms

The material in this section is not examinable

Knot theory is a branch of mathematics which arose from an attempt to classify knots rigorously by Peter Guthrie Tait (1831–1901). In 1920s, following the work of J. Alexander, knot theory became part of Topology.

A **knot** is a smooth injective function $S^1 \to \mathbb{R}^3$ where S^1 is the unit circle. The image of the function is a smooth closed curve in $\mathbb{R}^3$ which has no self-intersections; such curves are themselves called knots. Two knots K_1, K_2 are equivalent, or **isotopic,** if there is a continuous deformation of the space $\mathbb{R}^3$ which transforms K_1 into K_2.

The main problem of knot theory is to classify all knots up to isotopy. For example, a basic result of knot theory is that a **trefoil knot,** the curve shown in Figure 3.3, is not isotopic to the **unknot** (a straightforward copy of the circle in $\mathbb{R}^3$).

Is the problem of deciding whether two knots are isotopic related to the notion of homeomorphism? After all, every knot is homeomorphic to a circle, isn't it? Yes, but **complements** of knots are not homeomorphic. A highly non-obvious result is

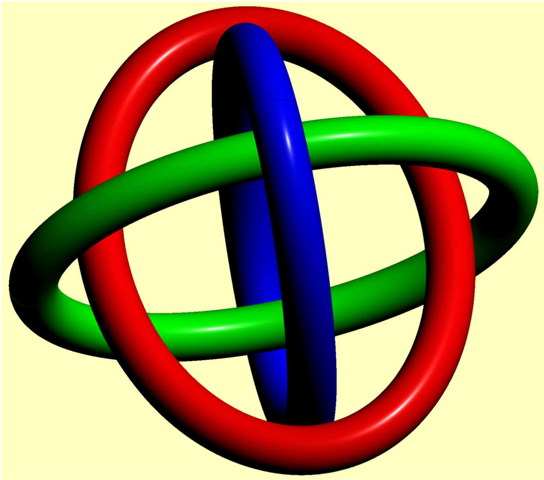

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Figure 3.4: *Borromean rings* [Link to online interactive 3D diagram]

Theorem: the Gordon-Luecke theorem (1989).

Two knots K_1, K_2 in $\mathbb{R}^3$ are isotopic (up to taking a mirror image) if, and only if, there exists a homeomorphism $\mathbb{R}^3 \setminus K_1 \xrightarrow{\sim} \mathbb{R}^3 \setminus K_2$.

The theory of knots extends to **links** which are unions of disjoint knots in $\mathbb{R}^3$. An example of a non-trivial link is the famous **Borromean rings:** three interlinked circles in $\mathbb{R}^3$, such that if any one circle is removed, the remaining two circles become unlinked, see Figure 3.4. Unlike knots, a link is not determined up to isotopy by a homeomorphism class of its complement.

A stylised representation of Borromean rings was chosen as the logo of the International Mathematical Union.

End of non-examinable material.

Topological properties

The following general type of a topological problem is of overwhelming importance in theory as well as applications.

Problem 1: the homeomorphism problem.

Given two topological spaces X, Y, determine whether X and Y are homeomorphic.

A homeomorphism $X \xrightarrow{\sim} Y$, if exists, can often be constructed explicitly as a map. Yet it may not be obvious how to justify a negative answer to Problem 1. The following concept is useful as it allows us to prove, in many cases, that two topological spaces are **not** homeomorphic.

Definition: topological property.

A property of topological spaces is called a **topological property** if, whenever a space X has this property, all spaces homeomorphic to X also have this property.

A standard approach to proving that two topological spaces X, Y are **not** homeomorphic is to find a topological property of X which is not shared by Y.

In simple cases, this can be achieved by going through a list of well-known topological properties and determining which of them X and Y have. We will now start putting together a (short) list of the most fundamental and important topological properties.

Among topological properties, we distinguish those which help us to solve the second main problem:

Problem 2: the continuous image problem.

Given two topological spaces X, Y, determine whether there exists a continuous surjective map $X \to Y$.

Of course, a negative answer to Problem 2 implies a negative answer to Problem 1. Problem 2 can be solved with the help of a topological property of X which must be shared by all continuous images of X and not just spaces homeomorphic to X. We will analyse each topological property to see if it helps us to solve Problem 2.

Hausdorff spaces

The first property of a topological space that we consider is being a Hausdorff space.

Definition: Hausdorff space.

A topological space X is **Hausdorff** if

$$\forall x, y \in X,\ x \neq y \implies \exists U, V \text{ open in } X\text{: } x \in U,\ y \in V,\ U \cap V = \emptyset.$$

In other words, **two distinct points of X must have disjoint open neighbourhoods.**

Proposition 3.1.

The Hausdorff property is a topological property.

Proof. Given topological spaces X, Y such that X is Hausdorff and $Y \underset{f}{\overset{\sim}{\to}} X$, we need to prove that Y is Hausdorff.

Let a, b be points of Y with $a \neq b$. The points $f(a), f(b)$ of X are distinct as f is injective. Hence, applying the definition of "Hausdorff" to X, we can find

- $V_{f(a)}$, $V_{f(b)}$ open in X such that $f(a) \in V_{f(a)}$, $f(b) \in V_{f(b)}$, $V_{f(a)} \cap V_{f(b)} = \emptyset$.

Put $U_a = f^{-1}(V_{f(a)})$ and $U_b = f^{-1}(V_{f(b)})$. Observe that

- U_a, U_b are open in Y because f is continuous so $f^{-1}(\text{open}) = \text{open}$;
- $a \in U_a$, $b \in U_b$;
- $U_a \cap U_b = f^{-1}(V_{f(a)} \cap V_{f(b)}) = f^{-1}(\emptyset) = \emptyset$.

We have thus verified the definition of "Hausdorff" for the space Y. □

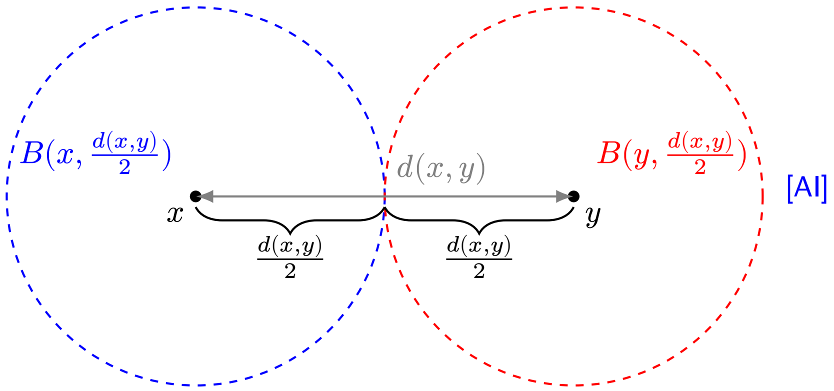

Figure 3.5: *"Metric topology is Hausdorff"*

Many "natural" examples of topological spaces are Hausdorff, for the following reason:

Proposition 3.2.

Metric topology is Hausdorff.

Proof. Let (X, d) be a metric space. Let x, y be points of X such that $x \neq y$; then, by axioms of a metric, the distance $d(x, y)$ is positive. Denote $r = \frac{d(x,y)}{2}$ and consider the open balls $U = B_r(x)$ and $V = B_r(y)$.

Then $U \ni x$, $V \ni y$, and a standard argument based on the triangle inequality shows that $U \cap V = \emptyset$. (See Figure 3.5 for an illustration.)

We have constructed disjoint open neighbourhoods of x, y and so we have verified the definition of "Hausdorff" for X. □

Proposition 3.3.

A subspace of a Hausdorff space is Hausdorff.

Proof. Let X be a Hausdorff topological space, and let A be a subset of X considered with the subspace topology.

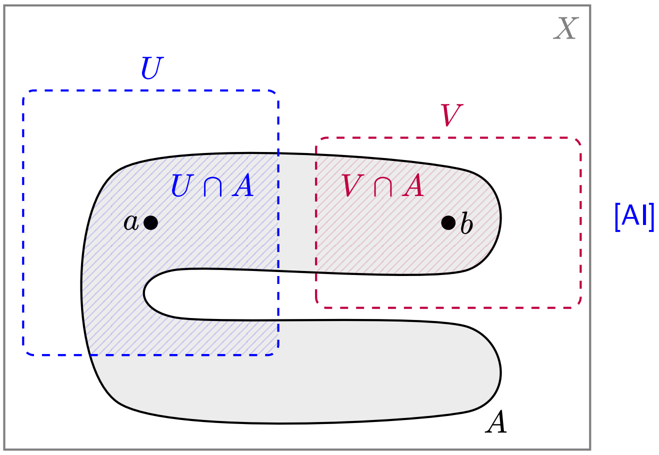

Figure 3.6: *"a subspace of a Hausdorff space is Hausdorff"*

Take two distinct points a, b of A. Then a, b are also distinct points of X, and so they have disjoint open neighbourhoods, $U \ni a$ and $V \ni b$, in X.

The sets U and V may not be subsets of A, so we put $U' = U \cap A$ and $V' = V \cap A$. Then

- $U' \ni a$ and $V' \ni b$;
- U', V' are open in A by definition of the subspace topology;
- $U' \cap V' = (U \cap A) \cap (V \cap A) = (U \cap V) \cap A = \emptyset \cap A = \emptyset$.

We have constructed disjoint open neighbourhoods of a, b in A and so we have verified the definition of "Hausdorff" for A. (See Figure 3.6 for an illustration.) □

Proposition 3.4.

In a Hausdorff space, a point is closed.

Proof. **Attention:** strictly speaking, a point is not a set and so cannot be closed. Still, "*a point is closed*" is a traditional shorthand for saying "*a set which consists of a single point is closed*", or, equivalently, "*a singleton is closed*": *singleton* means a one-point set.

Let $x \in X$ where X is a Hausdorff topological space. We need to show that the set $\{x\}$ is closed, equivalently that its complement $X \setminus \{x\}$ is open in X.

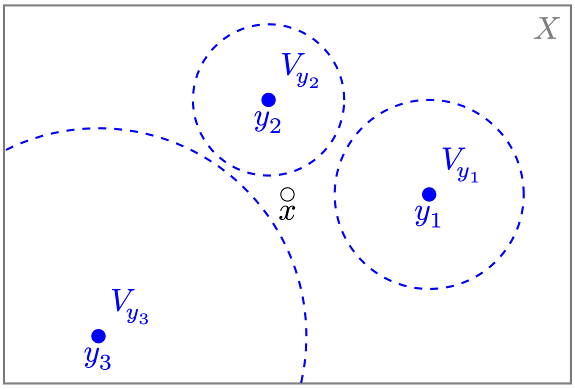

Figure 3.7: *the set $X \setminus \{x\}$ is the union of the open neighbourhoods V_y for all $y \neq x$*

For each point $y \in X \setminus \{x\}$, we have $y \neq x$ so by definition of "Hausdorff", there are open neighbourhoods $U_y \ni x$, $V_y \ni y$ which are disjoint: $U_y \cap V_y = \emptyset$. (Note that the set U_y depends on y, hence we subscript it with y even though it is an open neighbourhood of the point x.)

We are going to ignore $U_y(x)$ and only use the fact that V_y does not contain x. That is, $y \in V_y \subseteq X \setminus \{x\}$.

The set $\bigcup_{y \in X \setminus \{x\}} V_y$

- is open as a union of open sets V_y;
- is contained in $X \setminus \{x\}$, because $V_y \subseteq X \setminus \{x\}$ for each y;
- contains $X \setminus \{x\}$, because each point y of $X \setminus \{x\}$ is contained in the set V_y.

We conclude that $\bigcup_{y \in X \setminus \{x\}} V_y$ is, in fact, equal to $X \setminus \{x\}$. (See the illustration in Figure 3.7.) Therefore, we have proved that $X \setminus \{x\}$ is open. □

Do non-Hausdorff topological spaces exist? Yes, and they form an important class of topological spaces used in algebraic geometry (search: *Zariski topology*). We give a very simple example of a non-Hausdorff space.

Example: a non-Hausdorff topological space.

Show that the set $X = \{1, 2\}$ with the antidiscrete topology is not Hausdorff.

Solution: the only open sets in X are $\emptyset$ and $\{1, 2\}$, so the only open neighbourhood of the point 1 is $\{1, 2\}$. Also, the only only open neighbourhood of the point 2 is $\{1, 2\}$. Hence the distinct points 1 and 2 do not have disjoint open neighbourhoods, showing that X is not Hausdorff.

We note informally that the antidiscrete topology is the "weakest possible" topology as it has the fewest open sets, and there are not enough open sets to provide disjoint open neighbourhoods for points of the space. We will now formalise the notion of weaker (and stronger) topology.

Definition: stronger topology, weaker topology.

Suppose that $\mathscr{T}$ and $\mathscr{T}'$ are two topologies on the same set X. We say that the topology $\mathscr{T}'$ is **stronger** than $\mathscr{T}$ if every set, open in $\mathscr{T}$, is also open in $\mathscr{T}'$.

We say that $\mathscr{T}$ is **weaker** than $\mathscr{T}'$ if $\mathscr{T}'$ is stronger than $\mathscr{T}$. In summary,

$$\mathscr{T} \subseteq \mathscr{T}' \text{ means that } \mathscr{T} \text{ is weaker, } \mathscr{T}' \text{ is stronger.}$$

Exercise: let X be an arbitrary set.

1. Show that any topology on X is stronger than the antidiscrete topology on X and is weaker than the discrete topology on X.
2. Show that (X, discrete topology) is Hausdorff.
3. Show that (X, antidiscrete topology) is non-Hausdorff if, and only if, the cardinality of X is at least 2.

Proposition 3.5.

If $(X, \mathscr{T})$ is a Hausdorff topological space, and a topology $\mathscr{T}'$ on X is stronger than $\mathscr{T}$, then $(X, \mathscr{T}')$ is also Hausdorff.

Proof. Take any two points x, y in X such that $x \neq y$. Since $\mathscr{T}$ is Hausdorff, there exist

$\mathscr{T}$-open sets $U \ni x$, $V \ni y$ with $U \cap V = \emptyset$.

Since $\mathscr{T}'$ is stronger than $\mathscr{T}$, the sets U, V are open in $\mathscr{T}'$ as well. We have found disjoint $\mathscr{T}'$-open neighbourhoods of x, y and so we have verified the definition of "Hausdorff" for $\mathscr{T}'$. $\square$

References for the week 3 notes

Definition of **homeomorphism** is the same as [Sutherland, Definition 8.7]. Our claim that **'homeomorphic' is an equivalence relation** solves [Sutherland, Exercise 8.4].

Figure 3.1 is based on TikZ code written by OpenAI ChatGPT when asked to illustrate the homeomorphism between right open half-circle of the unit circle in the plane and the vertical straight line tangent to the half-circle at the point $(1, 0)$. The code improved by the AI as a result of feedback from YB on incorrect attempts (e.g., "some points that you are using are not on the right half-circle") and underwent minor visual re-styling by YB.

The **stereographic projection** $S^2 \setminus \{N\} \to \mathbb{R}^2$ is defined in [Armstrong, Figure 1.24]. **Figure 3.2** illustrating the stereographic projection is produced by code written by OpenAI ChatGPT in the Asymptote language intended for 3D technical drawing. Two earlier attempts were corrected by AI based on verbal feedback by YB.

The **Gordon-Luecke Theorem** (non-examinable) was proved in: C. M. A. Gordon and J. Luecke, Knots are determined by their complements, *J. Amer. Math. Soc.* **2** (1989), no. 2, 371–415.

The advanced subject of **knot theory** is touched upon in [Armstrong, Chapter 10]. **Figure 3.4,** showing the **link** known as **the Borromean rings** is produced by code written by OpenAI ChatGPT in the Asymptote language. Several incorrect diagrams were generated when the AI tried to represent the rings by perfect circles which is impossible. This diagram, which uses ellipses, was the response to feedback that the previous attempt was too visually complex.

We use the term **topological property** in the same sense as [Armstrong] and [Sutherland], but these textbooks do not formally define this term.

Figure 3.5 is TikZ code written by OpenAI ChatGPT to a prompt “illustrate the proof that metric topology is Hausdorff”.

Figure 3.6 is based on TikZ code by OpenAI ChatGPT to a prompt “illustrate the proof that a subspace of a Hausdorff space is Hausdorff”. Reworked by YB to add visual sophistication.

Proposition 3.2 (**metric implies Hausdorff**) is [Sutherland, Proposition 11.5].
Proposition 3.1 (**Hausdorffness is a topological property**) and Proposition 3.3 (**subspace of Hausdorff is Hausdorff**) solve [Sutherland, Exercise 11.4a,d].
Proposition 3.4 (**in Hausdorff, a point is closed**) solves [Sutherland, Exercise 11.2a].
The example **“antidiscrete $\{1, 2\}$ is not Hausdorff”** answers [Sutherland, Exercise 11.1].

[Sutherland, Definition 7.6] says “finer” and “coarser” in place of our **stronger** and **weaker** topology; we follow the terminology in [Willard] which is also used in Functional Analysis. Proposition 3.5 (**topology stronger than Hausdorff is Hausdorff**) elaborates on the remark made in [Willard, Example 13A.2].