Week 2

Exercises (answers at end)

Version 2024/11/02. To accessible online version of these exercises

Exercise 2.1. (a) Prove that the collection $\mathscr{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, +\infty) : x \in \mathbb{R}\}$ is a topology on the set $\mathbb{R}$ of real numbers.

(b) Prove that the collection $\mathscr{N} = \{\emptyset, \mathbb{R}\} \cup \{[x, +\infty) : x \in \mathbb{R}\}$ is not a topology on the set $\mathbb{R}$. Which axiom(s) of topology is/are not satisfied?

Exercise 2.2. Consider the set $X = \{1, 2\}$ with two points. Describe all possible topologies $\mathscr{T}$ on X. Among the topologies that you describe, identify the discrete topology, the antidiscrete topology and the cofinite topology.

Exercise 2.3. Call a subset A of $\mathbb{R}$ "cocountable" if $A = \emptyset$ or $\mathbb{R} \setminus A$ is finite or countably infinite.

(a) Show that the collection of all cocountable subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$.

(b) Is this topology the same as discrete topology? Antidiscrete? Cofinite topology?

Week 2

Exercises — solutions

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Exercise 2.1. (a) Prove that the collection $\mathscr{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, +\infty) : x \in \mathbb{R}\}$ is a topology on the set $\mathbb{R}$ of real numbers.

(b) Prove that the collection $\mathscr{N} = \{\emptyset, \mathbb{R}\} \cup \{[x, +\infty) : x \in \mathbb{R}\}$ is not a topology on the set $\mathbb{R}$. Which axiom(s) of topology is/are not satisfied?

Answer to E2.1. (a) **Axiom (i) of topology** requires that $\mathbb{R}$ is in $\mathscr{T}$. This axiom is satisfied.

Axiom (ii) of topology requires that the union, $\bigcup \mathscr{T}_1$, of every subcollection $\mathscr{T}_1$ of $\mathscr{T}$ be a member of $\mathscr{T}$. Let $\mathscr{T}_1$ be a subcollection of $\mathscr{T}$.

If $\mathscr{T}_1$ does not contain non-empty sets, then $\bigcup \mathscr{T}_1 = \emptyset$ which is a member of $\mathscr{T}$.

If $\mathscr{T}_1$ contains $\mathbb{R}$, then $\bigcup \mathscr{T}_1 = \mathbb{R}$ which is a member of $\mathscr{T}$.

The remaining possibility is that $\mathscr{T}_1$ is a family of sets $(x, +\infty)$ where x runs over some subset I of real numbers. Let $m = \inf I$. There are two cases:

- if $m = -\infty$, then $\bigcup \mathscr{T}_1 = \mathbb{R}$ (*justify this!*) which is in $\mathscr{T}$;

- if m is a finite real number, then $\bigcup \mathscr{T}_1 = (m, +\infty)$ (*justify this!*) which is also a member of $\mathscr{T}$.

In all cases, $\bigcup \mathscr{T}_1$ belongs to $\mathscr{T}$, so axiom (ii) is satisfied.

Axiom (iii) of topology requires that $A \cap B \in \mathscr{T}$ for any two sets $A, B \in \mathscr{T}$. Let $A, B \in \mathscr{T}$. The following case-by-case analysis shows that $A \cap B \in \mathscr{T}$:

A, B	$A \cap B$	$\in \mathscr{T}$?
$A = \emptyset$ or $B = \emptyset$	$\emptyset$	Yes
$A = \mathbb{R}$	B	Yes
$B = \mathbb{R}$	A	Yes
$A = (x, +\infty), B = (y, +\infty)$	$(\max(x, y), +\infty)$	Yes

(b) One can see that axioms (i) and (iii) are satisfied, but $\mathscr{N}$ fails axiom (ii). Indeed, consider the following infinite union of sets from $\mathscr{N}$:

$$\bigcup_{x>0} [x, +\infty) = (0, +\infty).$$

This union is not a set in $\mathscr{N}$. We have proved that $\mathscr{N}$ is not a topology.

Exercise 2.2. Consider the set $X = \{1, 2\}$ with two points. Describe all possible topologies $\mathscr{T}$ on X. Among the topologies that you describe, identify the discrete topology, the antidiscrete topology and the cofinite topology.

Answer to E2.2. There are four subsets of X: $\emptyset$, $\{1\}$, $\{2\}$ and X.

Let $\mathscr{T}$ be a topology. Axiom (i) of topology requires $X \in \mathscr{T}$, and Proposition 1.1 tells us that $\emptyset \in T$. This leaves us with four options, because we can either include or exclude each of the two *singleton sets*, $\{1\}$ and $\{2\}$. It is easy to see that **all four** collections are topologies on X:

- $\{\emptyset, X\}$: the antidiscrete topology.
- $\{\emptyset, \{1\}, X\}$.
- $\{\emptyset, \{2\}, X\}$.

- $\{\emptyset, \{1\}, \{2\}, X\}$: the collection of all subsets of X, that is, the discrete topology.

Note that for a finite set X, the discrete topology on X and the cofinite topology on X are the same thing. Indeed, in the cofinite topology, open sets are $\emptyset$ and all subsets of X with finite complement. Yet when X is finite, every subset of X has finite complement. Hence, for finite X, all subsets of X are open in cofinite topology, and all subsets of X are open in the discrete topology.

Exercise 2.3. Call a subset A of $\mathbb{R}$ "cocountable" if $A = \emptyset$ or $\mathbb{R} \setminus A$ is finite or countably infinite.

(a) Show that the collection of all cocountable subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$.

(b) Is this topology the same as discrete topology? Antidiscrete? Cofinite topology?

Answer to E2.3. (a) We modify the proof of Proposition 1.4 which deals with cofinite sets. We will change "finite" to "countable". By "countable" we mean "finite or countably infinite".

We need to prove that the collection $\mathscr{C}$ which consists of the empty set and all subsets of X with countable complement is a topology on the set X. Let us show that $\mathscr{C}$ fulfils axioms (i)–(iii) from the definition of topology.

(i) X has complement $\emptyset$, and $\emptyset$ is countable, so $X \in \mathscr{C}$.

(ii) Let $\mathscr{F}$ be some collection of sets from $\mathscr{C}$. If all sets in $\mathscr{F}$ are empty, then $\bigcup \mathscr{F} = \emptyset \in \mathscr{C}$.

Otherwise, take a non-empty set $U \in \mathscr{F}$. Then U must have countable complement, and $U \subseteq \bigcup \mathscr{F}$, so by lemma 1.2, $X \setminus \bigcup \mathscr{F} \subseteq X \setminus U$. Yet $X \setminus U$ is a countable set, and all subsets of a countable set are countable. Hence the complement of $\bigcup \mathscr{F}$ is countable, proving that $\bigcup \mathscr{F}$ is in $\mathscr{C}$.

(iii) Suppose $U, V \in \mathscr{C}$. If one of U, V is an empty set, then $U \cap V = \emptyset \in \mathscr{C}$.

Otherwise, U and V are non-empty, and since they are in $\mathscr{C}$, U and V must have countable complements. Then by the De Morgan laws $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$. Thus, $U \cap V$ has a countable complement (a union of two countable sets) and so $U \cap V \in \mathscr{C}$.

(b) We show that the cocountable topology on $\mathbb{R}$ is not the same as the discrete, the antidiscrete and the cofinite topologies. To show that two topologies are different, we exhibit a subset of $\mathbb{R}$ which is open in one topology but is not open in the other topology.

- the set $\mathbb{N}$ is not open in the cocountable topology ($\mathbb{R} \setminus \mathbb{N}$ is uncountable), yet $\mathbb{N}$ is open in the discrete topology (all sets are open in the discrete topology)
- the set $\mathbb{R} \setminus \mathbb{N}$ is open in the cocountable topology because its complement $\mathbb{N}$ is countable, yet $\mathbb{R} \setminus \mathbb{N}$ is not open in the antidiscrete topology (it is not one of $\emptyset$, $\mathbb{R}$) and is not open in the cofinite topology (its complement $\mathbb{N}$ is not finite).

References for the exercise sheet

The answer to E2.1(a) essentially solves [Sutherland, Exercise 7.6].

E2.2 is [Sutherland, Exercise 7.1a], and classification of topologies on $\{1, 2\}$ answers the second part of [Sutherland, Exercise 7.5].