Week 2

Metric topology. Open covers and bases. Subspace topology. Continuous functions

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Discrete, antidiscrete and cofinite topology on a set X, which we have introduced, are good for constructing simple counterexamples, yet they rarely lead to deep constructions in topology or help in applications of topology to other areas of mathematics and physics.

We will now connect abstract topology to the theory of metric spaces, studied in MATH21111. Metric spaces will provide us with a very rich class of examples of topological spaces.

Metric topologies. Euclidean topologies

Here is an **equivalent** definition of an open set given in MATH21111 *Metric spaces*.

> **Definition: open balls and open sets in a metric space**.
>
> Let (X, d) be a metric space. The **open ball** of radius $r > 0$ around a point $x \in X$ is the set $B_r(x) = \{y \in X : d(y, x) < r\}$.
>
> A d-**open set** in X is a union of open balls.

We quote a key result proved in MATH21111:

Theorem 2.1: metric-open sets in a metric space form a topology.

If (X, d) is a metric space, the collection $\mathscr{T}_d$ of all d-open sets in X is a topology on X. □

Topologies arising from metrics deserve a special definition:

Definition: metric topology, metrisable topological space.

The topology $\mathscr{T}_d$, where d is a metric on a set X, is called a **metric topology.**
A topological space $(X, \mathscr{T})$ is **metrisable,** if there exists a metric d on X such that $\mathscr{T} = \mathscr{T}_d$.

We now introduce what is arguably the most frequently used class of metric spaces and metric topologies. Let $n \geq 1$, and recall that $\mathbb{R}^n$ is the set of n-tuples $(x_1, \dots, x_n)$ of real numbers. The **Euclidean metric**

$$d_2((x_1, \dots, x_n), (y_1, \dots, y_n)) = \Big(\sum_{i=1}^{n} (x_i - y_i)^2\Big)^{1/2}$$

makes $\mathbb{R}^n$ a metric space.

Definition: Euclidean topology, Euclidean space.

The metric topology on $\mathbb{R}^n$ defined by the Euclidean metric is called the **Euclidean topology** and makes $\mathbb{R}^n$ the n-dimensional **Euclidean (topological) space.**

Furthermore, arbitrary subsets of $\mathbb{R}^n$ also become topological spaces:

Example: every subset of a Euclidean space is a topological space.

If $X \subset \mathbb{R}^n$, the Euclidean metric makes X a metric space hence a topological space.

We will study subspace topology in more detail in the next chapter.

A base of a topology. Open covers

The definition of an open set in a metric space (a union of open balls) motivates two notions which apply to arbitrary topologies.

Definition: open cover; base of a topology.

Let $(X, \mathscr{T})$ be a topological space.

An **open cover** of X is a collection $\mathscr{C}$ of open subsets of X whose union is X: $\bigcup \mathscr{C} = X$.

A **base** of the topology $\mathscr{T}$ is a collection $\mathscr{B}$ of subsets of X such that $\mathscr{T}$ consists of unions of all subcollections of $\mathscr{B}$.

Remark 1: every topology has at least one base. For example, the whole collection $\mathscr{T}$ is a base for $\mathscr{T}$. But smaller bases are usually more interesting.

Remark 2: a base $\mathscr{B}$ of the topology on $(X, \mathscr{T})$ must be an open cover. Indeed, for every set $U \in \mathscr{B}$, the union $\bigcup\{U\}$ of the single-set subcollection $\{U\}$ of $\mathscr{B}$ must belong to $\mathscr{T}$. That is, every set $U \in \mathscr{B}$ is an open set.

Moreover, $X \in \mathscr{T}$ by axiom (i) of topology, and $\mathscr{B}$ is a base, so X must be a union of some subcollection of $\mathscr{B}$. Hence $\bigcup \mathscr{B} \supseteq X$. On the other hand, a union of subsets of X is a subset of X, so $\bigcup \mathscr{B} \subseteq X$. Thus, $\bigcup \mathscr{B} = X$.

We have shown that all sets in $\mathscr{B}$ are open, and that the union of all sets in $\mathscr{B}$ is X. Therefore, $\mathscr{B}$ is an open cover of X.

Remark 3: although every base is an open cover, not every open cover is a base. For example, the Euclidean space $\mathbb{R}^n$ has open cover $\{\mathbb{R}^n\}$, but the only unions of subcollections of this cover are $\emptyset$ and $\mathbb{R}^n$. This does not exhaust open subsets of $\mathbb{R}^n$, which has infinitely many open subsets.

Example 2.2: open balls form a base of a metric topology.

By definition, a metric topology $\mathscr{T}_d$ has base $\mathscr{B} = \{\text{all open balls in the metric } d\}$.

Does the Euclidean topology on $\mathbb{R}^n$ have other bases? Yes, plenty of other bases are possible.

First of all, we note that a metric d which defines a given metrisable topology on X may not be unique. Recall from MATH21111 that two metrics d and e on X are **topologically equivalent** if d-open sets are the same as e-open sets; that is, $\mathscr{T}_d = \mathscr{T}_e$. We omit the proof of the following result, which can be found in MATH21111 or in the literature.

Proposition 2.3: Lipschitz equivalent metrics are topologically equivalent.

Suppose that the metrics d and e on a set X are **Lipschitz equivalent**, that is, there are two positive numbers k and k such that

$$\forall x, y \in X, \quad he(x, y) \leq d(x, y) \leq ke(x, y).$$

Then d and e are topologically equivalent metrics. □

Note that this is **not** an if-and-only-if result: there may be metrics on X which are not Lipschitz equivalent yet are topologically equivalent.

Remark: it was shown in MATH21111 *Metric spaces* that the Euclidean metric d_2 on $\mathbb{R}^n$ is Lipschitz equivalent to d_1 (the "Manhattan metric" or the "taxicab metric") defined by

$$d_1((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^{n} |x_i - y_i|,$$

and also to the metric d_∞, defined by

$$d_\infty((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max_{i=1}^{n} |x_i - y_i|.$$

In fact, for all $x, y \in \mathbb{R}^n$,

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq nd_\infty(x, y),$$

which shows that d_1, d_2 and d_∞ are pairwise Lipschitz equivalent. By Proposition 2.3, d_1, d_2 and d_∞ define the same topology on $\mathbb{R}^n$ — the Euclidean topology.

However, Example 2.2 tells us that each of the three metrics defines a **base** for the Euclidean topology; the base consists of open balls of arbitrary radii around each point in the

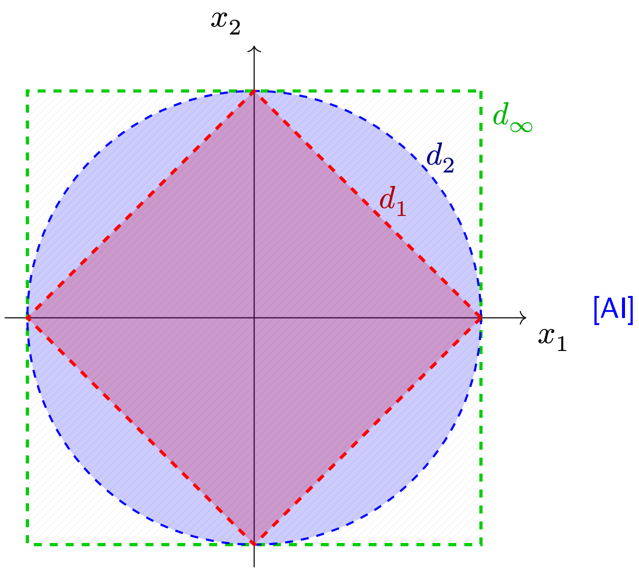

Figure 2.1: *Each of the three metrics d_1, d_2, d_∞ on $\mathbb{R}^2$ defines its own open unit ball around $(0,0)$. The three unit balls are shown on the same diagram for ease of comparison.*

plane. We note that these balls are of different shape. Figure 2.1 shows the d_1-open ball, the d_2-open ball and the d_∞-open ball in $\mathbb{R}^2$, of radius 1, centred at the same point. We clearly have three different bases for the same Euclidean topology on $\mathbb{R}^2$:

- a base which consists of all open d_1-rhombuses around each point;
- a base which consists of all open d_2-discs around each point;
- a base which consists of all open d_∞-squares around each point of the plane.

There are, of course, infinitely many more bases for the Euclidean topology on $\mathbb{R}^2$.

Continuous functions

One of the reasons to introduce a topological space as a more general structure than a metric space is to be able to define continuous functions without a metric.

Let X, Y be sets, initially considered without a topological space structure. We write

$f\colon X \to Y$ to denote a function with domain X and codomain Y. The words **function, map, mapping** will mean the same thing. The following notation and terminology will be used.

Definition: image of an element, image of a set, preimage of a set.

Let $f\colon X \to Y$ be a function. For $x \in X$, the element $f(x)$ of Y is the **image of** x under f. For a subset $A \subseteq X$, the subset of Y defined as

$$f(A) = \{f(a) : a \in A\}$$

is the **image of the set** A under f. For a subset $B \subseteq Y$, the subset of X defined as

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

is called **the preimage of the set** B under f.

We are now going to define **continuity,** which does require a topology on both X and Y.

Definition: continuous function.

Let X and Y be topological spaces. A function $f\colon X \to Y$ is **continuous** if for every subset V of Y, open in Y, the preimage $f^{-1}(V)$ is open in X.

Remark: in MATH21111 *Metric Spaces,* this was shown to be an equivalent definition of continuity. This means that when X and Y are metric spaces, considered with their metric topologies, we can use **two** equivalent definitions of a continuous function:

- the ε - δ definition of continuity for functions between metric spaces;
- the topological definition of continuity, given above.

When X or Y is **not** a metric space, we do **not** have the ε - δ definition, and can only use the topological definition.

Warning: remember,

"f **is continuous**" means $f^{-1}($ **open** $) =$ **open !**

Figure 2.2: *A subset of* $\mathbb{R}$ *can be open & closed, open, closed, or neither*

It is **not true** for a general continuous function that **images** of open subsets of X are open in Y: $f($ open $) \neq$ open !!!

It is often useful to characterise continuous functions in terms of **closed sets,** which we will now define.

Closed sets

> **Definition: closed set.**
>
> Let X be a topological space. A subset F of X is **closed** in X if its complement $X \setminus F$ is open in X.

This definition means that every subset A of a topological space X falls into one of the four classes:

- A is open and closed; for example, $A = \emptyset$ or $A = X$;
- A is open but not closed; for example, $X = \mathbb{R}$ (Euclidean topology), $A = (0, 1)$;
- A is closed but not open; for example, $X = \mathbb{R}$ (Euclidean topology), $A = [2, 3]$;
- A is neither open nor closed; for example, $X = \mathbb{R}$ (Euclidean topology), $A = [4, 5)$.

The last three examples are illustrated by Figure 2.2.

> **Alert.**
>
> "Not open" does not mean "closed"!

The collection of closed sets in X has properties which mirror, but do not repeat, the properties of open sets – we need to exchange unions and intersections:

Proposition 2.4: properties of closed sets.

If X is a topological space, then

(a) $\emptyset$ and X are closed in X,

(b) arbitrary intersections of closed sets are closed,

(c) finite unions of closed sets are closed.

Proof. Since closed sets are complements of open sets, these properties follow by applying the De Morgan laws to the properties of open sets in Proposition 1.1. □

Proposition 2.5: the closed set criterion of continuity.

A function $f\colon X \to Y$ between topological spaces X and Y is continuous if, and only if, the preimage of every closed subset of Y is closed in X.

Proof. The key point of the proof is the following property of preimages:

$$\forall V \subseteq Y, \quad f^{-1}(Y \setminus V) = X \setminus f^{-1}(V),$$

in other words, **the preimage of a complement is the complement of the preimage.** (See the week 1 tutorial where this was discussed.)

Assume that $f\colon X \to Y$ is continuous. Let $F \subseteq Y$ be any closed set in Y. Then $V = Y \setminus F$ is open in Y. We compute $f^{-1}(F)$ as follows: $f^{-1}(F) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Since $f^{-1}(V)$ is open (by continuity of f), its complement $X \setminus f^{-1}(V)$ is closed, as required. We proved that the preimage of every closed set is closed.

Now assume that the preimage of every closed set under f is closed. Let $V \subseteq Y$ be any open set in Y. Then the complement $Y \setminus V$ of V is closed in Y, so, by assumption, $f^{-1}(Y \setminus V)$ is closed in X. Yet $f^{-1}(Y \setminus V)$ equals $X \setminus f^{-1}(V)$ and, since this set is closed, $f^{-1}(V)$ must be open. We proved that the preimage of every open set is open, and so we have verified the definition of "continuous" for f. □

Easy properties and examples of continuous functions

Unlike in Mathematical Foundations and Analysis, in general we cannot form "sums" or "products" of continuous functions from X to Y because the topological space Y may not have any $+$ or $\times$ operations defined on it. Yet we may form compositions:

Proposition 2.6: composition of continuous functions is continuous.

Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ where X, Y, Z are topological spaces, and the functions f, g are continuous. Then the composition $X \xrightarrow{g \circ f} Z$ is also continuous.

Proof. The key point of the proof is the formula for the **preimage under composition,** left as an exercise to the student:

$$\forall W \subseteq Z, \quad (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)).$$

Let $W \subseteq Z$ be any open subset of Z. Since g is continuous, $g^{-1}(W)$ is open in Y. Since f is continuous, $f^{-1}(g^{-1}(W))$ is open in X. We have proved that $(g \circ f)^{-1}(W)$ is open in X, and so we have verified the definition of "continuous" for $g \circ f$. □

Example: the identity map is continuous.

Let X be a topological space. Show that the **identity map on** X, $\mathrm{id}_X \colon X \to X$ defined by $x \mapsto x$ for all points $x \in X$, is continuous.

Solution: for any V open in X, the preimage $\mathrm{id}_X^{-1}(V) = V$ is open. This proves that id_X is a continuous function.

Example: a constant function is continuous.

Let X, Y be topological spaces and let y_0 be a point of Y. A **constant function** is a function of the form $\mathrm{const}_{y_0} : X \to Y$, $x \mapsto y_0$ for all $x \in X$ (i.e., the function that sends the whole of X to one point). Show that constant functions are continuous.

Solution: if V is open in Y, the preimage of V under a constant function is as follows:

$$\mathrm{const}_{y_0}^{-1}(V) = \begin{cases} X, & \text{if } y_0 \in V, \\ \emptyset, & \text{if } y_0 \notin V. \end{cases}$$

As X and $\emptyset$ are open in X, the preimage of V is always open. Continuity is proved.

Subspace topology. The inclusion map in_A

Every subset of a topological space is made a topological space in its own right, as follows.

> **Definition: subspace topology.**
>
> Let X be a topological space and let A be a subset of X.
>
> A subset $V \subseteq A$ is called **open in** A if there exists $U \subseteq X$ such that U is open in X and $V = U \cap A$.
>
> The collection $\mathscr{T}_A$ of subsets of A open in A is a topology on A, called the **subspace topology.** By a **subspace** of a topological space X we mean a space $(A, \mathscr{T}_A)$.

The definition of "open in A" is illustrated by Figure 2.3. Strictly speaking, we need to prove that a "subspace topology" is indeed a topology. We do not go through the proof, given below, in class, and it is often left as an exercise in the literature.

> **Example: subspace topology is indeed a topology.**
>
> Let X be a topological space, and let $A \subseteq X$. Show that $\mathscr{T}_A$ is a topology on A.

Solution (*not given in class*): Axiom (i) of topology requires $A \in \mathscr{T}_A$. We have $A = X \cap A$ where the set X is open in X, hence by definition of "open in A", $A \in \mathscr{T}_A$, as required.

Axiom (ii) requires that the union of any subcollection of $\mathscr{T}_A$ be again in $\mathscr{T}_A$. Let $\{V_\alpha : \alpha \in I\}$ be a subcollection of $\mathscr{T}_A$. Then for every α, V_α is open in A, and so there exists a set U_α open in X such that $V_\alpha = U_\alpha \cap A$. We have

$$\bigcup_{\alpha\in I} V_\alpha = \bigcup_{\alpha\in I}\Big(U_\alpha \cap A\Big) = \Big(\bigcup_{\alpha\in I} U_\alpha\Big) \cap A$$

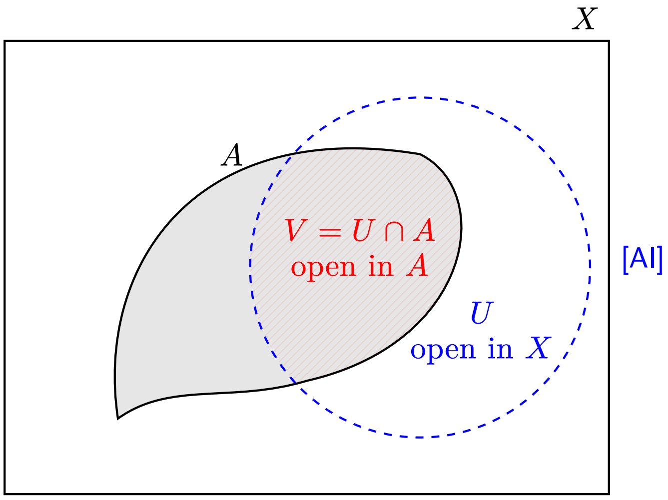

Figure 2.3: *meaning of "a subset V of A is open in A".*

(the last step is a known distributive law for $\cap$ and $\cup$). The set $\bigcup_{\alpha\in I} U_\alpha$ is open in X since it is a union of open sets, so by definition of "open in A", $\bigcup_{\alpha\in I} V_\alpha \in \mathscr{T}_A$, as required.

Axiom (iii) of topology requires that if $V, V' \in \mathscr{T}_A$ then $V \cap V' \in \mathscr{T}_A$. Let $V, V' \in \mathscr{T}_A$. Then there exist sets U, U', open in X, such that $V = U \cap A$ and $V' = U' \cap A$. Then $V \cap V' = (U \cap A) \cap (U' \cap A) = (U \cap U') \cap A$. The set $U \cap U'$ is open in X as an intersection of two open sets, so, by definition of "open in A", $V \cap V' \in \mathscr{T}_A$, as required.

To each subspace of X there is associated a continuous map:

Definition: inclusion map.

Let A be a subspace of a topological space X. The function $\mathrm{in}_A \colon A \to X$, defined by $a \mapsto a$ for all $a \in A$, is the **inclusion map** of A.

If $A = X$, we have $\mathrm{in}_X = \mathrm{id}_X$. It turns out that the inclusion map is always continuous:

Proposition 2.7: the inclusion map is continuous.

If A is a subspace of a topological space X, the inclusion map $\mathrm{in}_A \colon A \to X$ is continuous.

Proof. Let U be an arbitrary open subset of X. The preimage

$$\mathrm{in}_A^{-1}(U) = \{a \in A : \mathrm{in}_A(a) \in U\} = \{a \in A : a \in U\} = U \cap A$$

is open in A by definition of subspace topology. Continuity of in_A is proved. □

References for the week 2 notes

The Euclidean space $\mathbb{R}^n$ is written as $\mathbb{E}^n$ in [Armstrong].

A **base** of a topology is defined in the same way in [Armstrong, Section 2.1] but is called a **basis** in [Sutherland, Definition 8.9].

Proposition 2.3 that **Lipshitz equivalence implies topological equivalence** is [Sutherland, Proposition 6.34]. Metrics d_1, d_2 and d_∞ were introduced in MATH21111. They are also defined for $n = 2$ in [Sutherland, Example 5.7].

Figure 2.1 is based on LaTeX/TikZ code generated by OpenAI ChatGPT in response to the following prompt by YB given below. YB made minor edits to the code to improve visual appearance.

> Can you produce LaTeX or TikZ code which would generate drawing showing, in the same pair of coordinate axes, the image of the d_1-unit ball around the origin, the d_2-unit ball around the origin, and the d_∞-unit ball around the origin in the plane $\mathbb{R}^2$? The three unit balls must be of different color. Here d_1 denotes the "Manhattan metric" on the plane, d_2 is the Euclidean metric, and d_∞ is the metric where the distance between the points (x_1, x_2) and (y_1, y_2) is defined as $\max(|x_1 - x_2|, |y_1 - y_2|)$.

The definition of a **continuous function** via preimages of open sets is standard in topology, see [Sutherland, Definition 8.1]. However, [Armstrong] uses a different definition, shown to be equivalent to ours in [Armstrong, Theorem (2.6)]. In this course, we do not need the notion "f is continuous at a point x": interested students can check [Sutherland, Definition 8.2].

Closed sets are defined in [Sutherland, Definition 9.1], and our Proposition 2.4 is [Sutherland, Proposition 9.4]. Our **closed set criterion of continuity,** Proposition 2.5, is [Sutherland, Proposition 9.5], yet Sutherland omits the proof. Proposition 2.6, **continuity of composition,** is [Sutherland, Proposition 8.4]. Our examples showing that id_X **is continuous** and **constants are continuous** solve [Sutherland, Exercise 8.1(a,b)].

Figure 2.3 is based on TikZ code generated by OpenAI ChatGPT when asked to illustrate the definition of subspace topology. The area of V to be shaded was calculated incorrectly by AI, and YB replaced the calculation with a call to the TikZ `clip` function call.

Our definition of **subspace topology** is [Sutherland, Definition 10.3], and the proof that it is a topology solves [Sutherland, Exercise 10.2]. Proposition 2.7, **the inclusion map is continuous,** is [Sutherland, Proposition 10.4].