Week 1

Topology: basic definitions and examples

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The year-long MATH31010 Topology and Analysis course will consist of **three parts:**

I. Introduction to Topology, lectured by Yuri Bazlov;
II. Introduction to Functional Analysis, lectured by Yotam Smilansky;
III. Further topics in topology and analysis, lectured by Donald Robertson.

You are reading **Part I notes** which are being developed to reflect the content of the course as taught in the 2024/25 academic year. Questions and comments on these lecture notes should be directed to Dr Yuri Bazlov at `Yuri.Bazlov@manchester.ac.uk`.

Textbooks: some proofs will follow the book [Sutherland] or [Armstrong], see comments at end of each chapter. Overall organisation of the material differs from either book.

Use of generative AI in these notes: by way of an experiment, some diagrams in these notes will use source code generated with the help of artificial intelligence (AI). An acknowledgment will be provided via an [AI] link next to the diagram.

AI is an evolving set of technologies which utilise applications of pure mathematics, including topology (example: topological data analysis). It seems especially fitting that generative AI can now help us visualise definitions and proofs from the Topology and Analysis course.

An informal overview

Many processes in nature and in industry are modelled by continuous functions. The notion of "continuous" was defined for functions $f\colon X \to Y$ where

- X, Y are subsets of $\mathbb{R}$ (B. Bolzano, A.-L. Cauchy, first half of the 19$^{\text{th}}$ century) — as discussed in MATH11121 *Mathematical Foundations and Analysis*;
- X, Y are metric spaces (M. Fréchet, early 20$^{\text{th}}$ century) — as discussed in MATH21111 *Metric Spaces*.

Yet some mathematical situations expect continuous functions defined between

- sets with a large class of metrics and no single preferred metric, or
- sets where no metric exists.

An example of a set where no natural metric may exist is an algebraic curve, or more generally an algebraic variety, over a field other than $\mathbb{R}$ or $\mathbb{C}$. Algebraic geometry, a branch of mathematics which was expanding and becoming more formal in early 20$^{\text{th}}$ century, required a rigorous theory of continuous functions between such objects. A nice class of algebraic curves are elliptic curves, which today have extensive applications in many fields, including number theory and cryptography.

Thus, **an important goal of topology is to define "continuous" without a metric.** For this, the sets X and Y need to be equipped with a structure of a **topological space** defined by purely set-theoretic axioms. This approach was developed in the 20$^{\text{th}}$ century work of F. Riesz and F. Hausdorff. (We give the formal definitions after this introduction.) Every metric space is automatically a topological space — you had a glimpse of that in MATH21111 Metric Spaces, where the notion of an *open set* was introduced — but there are topological spaces which cannot be defined by means of a metric.

Two topological spaces X, Y are considered equivalent, or **homeomorphic**, if there are bijective functions $f\colon X \to Y$ and $g\colon Y \to X$ which are inverse to each other and are both continuous. This notion is new even for metric spaces: two metric spaces might have very different metrics, but still be homeomorphic. A simple example, see Figure 3.1, shows that the real line $\mathbb{R}$ is homeomorphic to its subset, the open interval $(-\pi/2, \pi/2)$. Yet $\mathbb{R}$ is complete and unbounded metric space, whereas $(-\pi/2, \pi/2)$ is bounded but not complete.

Topology is concerned with finding properties of a topological space X which are necessarily shared by all spaces homeomorphic to X. Such properties are called **topological properties** for short. The above example tells us that boundedness, and completeness, are **not** topological properties of a metric space.

Especially sought after are topological properties which persist in any continuous image of the space X: that is, $f(X)$ where f is continuous, but f is not necessarily a homeomorphism (i.e., f may not be injective, or the inverse of f may not be continuous). In this course, we will study two such important properties: "X is compact" and "X is connected".

The result which states "if X is compact and f is continuous, then $f(X)$ is compact" is a vast generalisation of the theorem from Mathematical Foundations and Analysis which says that a continuous function on a closed bounded interval in $\mathbb{R}$ is bounded and attains its maximum and minimum value. (This was mentioned in MATH21111 Metric Spaces.) We can use this result in many more situations, for example to show that there is no surjective continuous map from a sphere to $\mathbb{R}^2$.

The result which states "if X is connected and f is continuous, then $f(X)$ is connected" is, in turn, a generalisation of the Intermediate Value Theorem from Mathematical Foundations and Analysis. Again, we can apply it to many more situations, for example to give a rigorous proof that the shapes $\bigcirc$ (circle) and ∞ (figure of eight) are not homeomorphic.

But the true power of topology lies in its ability to apply the same principles to simple spaces (the circle, $\mathbb{R}^2$ etc) and to vastly more complicated objects such as infinite dimensional normed spaces. Hopefully you will see some of the workings of topology in infinite dimensions in parts II and III of the course.

Fundamentals of sets

Definitions and axioms in topology are expressed in the language of set theory. We need to be able to speak this language. In this section, we recall fundamental notions from set theory and introduce some notation to be used throughout the course.

> **Notation: set, element, collection.**
>
> **Sets** will be denoted by capital letters A, B, C, ...
>
> **Elements** of a set will typically be written as small letters: $a, b, c \in A$ means that a, b and c are elements of the set A.
>
> A **collection** (= **family**) of sets is a finite or infinite list of sets. Collections will be denoted by script letters such as $\mathscr{E}$, $\mathscr{F}$, $\mathscr{G}$, ... Sets in a collection may be indexed by some index set: for example, $\mathscr{F} = \{A_\alpha : \alpha \in I\}$ where I is an index set.

It is important to distinguish between elements of a set A and subsets of a set A. Recall that B is a **subset** of A if every element of B is also an element of A:

$$B \subseteq A \quad \overset{\text{def}}{\iff} \quad \forall x \in B,\ x \in A.$$

We will also consider **subcollections:** a collection $\mathscr{G}$ of sets may be a subcollection of a collection $\mathscr{F}$.

While sets, subsets and elements appear in mathematics of all levels and styles, collections (especially infinite collections) of sets tend to occur in advanced pure mathematical texts. To familiarise ourselves with collections, let us look at simple examples built from subsets of $\mathbb{R}$:

- $\mathscr{F} = \{A\}$, a collection of just one set.
- $\mathscr{G} = \{A, A\}$, a collection of two identical sets. We do not insists that all sets in a collection are different from each other — repetitions are allowed.
- $\mathscr{C} = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$, a countable collection of open intervals in $\mathbb{R}$.
- $\mathscr{U} = \{(0, x) : x \in \mathbb{R},\ x > 0\}$, an uncountable collection of open intervals in $\mathbb{R}$.
- $\mathscr{E} = \emptyset$, an empty collection of subsets of $\mathbb{R}$.

Two common operations can be applied to a collection of subsets of some universal set X:

Definition: union and intersection of a collection.

Let X be a set and $\mathscr{F}$ be a collection of subsets of X. The **union** of $\mathscr{F}$ is the set

$$\bigcup \mathscr{F} = \{x \in X : \exists A \in \mathscr{F},\ x \in A\}.$$

The **intersection** of $\mathscr{F}$ is the set

$$\bigcap \mathscr{F} = \{x \in X : \forall A \in \mathscr{F},\ x \in A\}.$$

When a collection is finite, or when it is indexed by an index set, alternative notation is often used for the union and intersection of the collection.

Notation: variants of notation for unions and intersections.

$\mathscr{F} = \{A, B\}$	$\Rightarrow$	we can write $\bigcup \mathscr{F} = A \cup B$.
$\mathscr{F} = \{A_1, A_2, \dots, A_n\}$, a collection of n sets	$\Rightarrow$	$\bigcup \mathscr{F} = A_1 \cup A_2 \cup \cdots \cup A_n$.
$\mathscr{F} = \{A_i : i \in \mathbb{N}\}$, a countable collection	$\Rightarrow$	$\bigcup \mathscr{F} = \bigcup_{i=1}^{\infty} A_i$.
$\mathscr{F} = \{A_\alpha : \alpha \in I\}$ for some index set I	$\Rightarrow$	$\bigcup \mathscr{F} = \bigcup_{\alpha \in I} A_\alpha$.

The above conventions also apply to $\bigcap$ *in place of* $\bigcup$.

For practice, we work out the unions and intersections of collections of subsets of $\mathbb{R}$ in the example above. We use the notation $\mathbb{R}_{>0}$ for the set of all positive reals.

- $\mathscr{F} = \{A\}$ where $A \subseteq \mathbb{R}$: $\bigcup \mathscr{F} = \bigcap \mathscr{F} = A$.
- $\mathscr{G} = \{A, A\}$: $\bigcup \mathscr{G} = \bigcap \mathscr{G} = A$.
- $\mathscr{C} = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$: $\bigcup_{n=1}^{\infty} (0, \frac{1}{n}) = (0, 1)$, $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$.
- $\mathscr{U} = \{(0, x) : x \in \mathbb{R}_{>0}\}$: $\bigcup_{x>0} (0, x) = \mathbb{R}_{>0}$, $\bigcap_{x>0} (0, x) = \emptyset$.
- $\mathscr{E} = \emptyset$, an empty collection of subsets of $\mathbb{R}$: $\bigcup \mathscr{E} = \emptyset$, $\bigcap \mathscr{E} = \mathbb{R}$.

Note that in the last example, the intersection of an empty collection of subsets of X is X, the universal set. This follows logically, since the condition "$\forall A \in \emptyset$, $x \in A$" in the definition of intersection holds for all x.

Another way to see that an empty collection has intersection equal to the universal set is to use the De Morgan laws. We will recall the De Morgan laws soon.

Topology: definition and examples. Open sets

Although the word **"topology"** may mean the area of mathematics studied in this course, we say **"a topology"** to refer to a collection of sets described in the following definition.

Definition: a topology; topological space; point; open set.

Let X be a set. Suppose $\mathscr{T}$ is a collection of subsets of X such that

(i) $X \in \mathscr{T}$;

(ii) for every subcollection $\mathscr{T}_1$ of $\mathscr{T}$, the set $\bigcup \mathscr{T}_1$ belongs to $\mathscr{T}$;

(iii) if $A \in \mathscr{T}$ and $B \in \mathscr{T}$, then $A \cap B \in \mathscr{T}$.

Then $\mathscr{T}$ is called **a topology** on the set X, and the pair $(X, \mathscr{T})$ is a **topological space.** Elements of X are called **points.** Subsets of X which belong to the topology, i.e., belong to the collection $\mathscr{T}$, are called **open sets.**

We may say "X is a topological space" instead of "$(X, \mathscr{T})$ is a topological space" if it is clear which topology $\mathscr{T}$ is used. (Keep in mind that more than one topology may be defined on the same set X.)

We arrive at our first explicit (but trivial!) example of a topological space. It is easy to see that axioms (i)–(iii) hold, as all unions and intersections are equal to the empty set:

Example: topology on the empty set.

Let $X = \emptyset$, the empty set. Consider the collection $\mathscr{T} = \{\emptyset\}$ — this collection consists of just one set. The pair $(\emptyset, \{\emptyset\})$ is a topological space.

The following properties of open sets are an easy consequence of the definition of topology. In fact, these properties are equivalent to the axioms of topology:

Proposition 1.1: properties of open sets.

If $(X, \mathscr{T})$ is a topological space, then

(a) X and $\emptyset$ are open,

(b) arbitrary unions of open sets are open,

(c) finite intersections of open sets are open.

Sketch of proof. (a) X is open by axiom (i); $\emptyset$ is the union of an empty collection of open sets so is open by axiom (ii).

(b) is just axiom (ii) of topology.

For (c), we need to assume that $U_1, U_2, \dots, U_n$ are open sets, and to show that $U_1 \cap \dots \cap U_n$ is open. This is shown by induction where the base case $n = 2$ is axiom (iii) of topology, and the inductive step is done by writing

$$U_1 \cap \dots \cap U_n = (U_1 \cap U_2) \cap U_3 \cap \dots \cap U_n$$

as the intersection of $n - 1$ open sets. □

Alert: arbitrary intersections of open sets.

Only intersections of **finitely many** open sets are guaranteed to be open. The intersection of an **infinite** collection of open sets may not be open. Counterexamples will be seen in subsequent lectures and in the tutorial.

In general, there exist many topologies on a given set X. We give simple examples of topologies which can be introduced on any set.

Definition: discrete, antidiscrete and cofinite topologies.

Let X be a set.

- The **discrete topology** on X is the topology where all subsets of X are open.
- The **antidiscrete topology** on X is where the only open sets are $\emptyset$ and X.
- The **cofinite topology** on X: $U \subseteq X$ is open iff $U = \emptyset$ or $X \setminus U$ is a finite set.

Exercise. Show that the discrete topology is a topology. That is, the collection of **all** subsets of X satisfies axioms (i)–(iii) of topology. Do the same for the antidiscrete topology.

To show that the cofinite topology is a topology, we carefully work with complements.

Definition: complement.

Let X be a set and A be a subset of X. The subset $X \setminus A$ of X is called the **complement** of the set A in X.

For the next result, we need a simple fact about complements (proof: exercise).

Lemma 1.2: taking the complement reverses inclusion.

If A, B are subsets of X, then $A \subseteq B \iff (X \setminus B) \subseteq (X \setminus A)$. □

We also need the De Morgan laws (proof: exercise). The proof of the next Lemma is an exercise and can be found in the literature.

Lemma 1.3: the De Morgan laws.

Let $\{A_\alpha : \alpha \in I\}$ be an arbitrary family of subsets of X. Then
$X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$ and $X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$.

(Note how $\bigcup$ changes to $\bigcap$ and vice versa.) In short, the De Morgan laws say

the complement of a union is the intersection of complements,
and
the complement of an intersection is the union of complements.

We now use lemma 1.2 and the De Morgan laws to prove

Proposition 1.4: the cofinite topology is a topology.

The collection $\mathscr{C}$ which consists of the empty set and all subsets of X with finite complement is a topology on the set X.

Proof. We show that $\mathscr{C}$ satisfies axioms (i)–(iii) from the definition of topology.

(i) X has complement $\emptyset$, and $\emptyset$ is finite, so $X \in \mathscr{C}$.

(ii) Let $\mathscr{F}$ be some collection of sets from $\mathscr{C}$. If all sets in $\mathscr{F}$ are empty, then $\bigcup \mathscr{F} = \emptyset \in \mathscr{C}$.

Otherwise, take a non-empty set $U \in \mathscr{F}$. Then U must have finite complement, and $U \subseteq \bigcup \mathscr{F}$, so by lemma 1.2, $X \setminus \bigcup \mathscr{F} \subseteq X \setminus U$. Yet $X \setminus U$ is a finite set, and all subsets of a finite set are finite. Hence the complement of $\bigcup \mathscr{F}$ is finite, proving that $\bigcup \mathscr{F}$ is in $\mathscr{C}$.

(iii) Suppose $U, V \in \mathscr{C}$. If one of U, V is an empty set, then $U \cap V = \emptyset \in \mathscr{C}$.

Otherwise, U and V are non-empty, and since they are in $\mathscr{C}$, U and V must have finite complements. Then by the De Morgan laws $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$. Thus, $U \cap V$ has a finite complement (a union of two finite sets) and so $U \cap V \in \mathscr{C}$. □

References for the week 1 notes

Both [Armstrong] and [Sutherland] use the terms **collection** and **family** interchangeably.

Our definition of a **topological space** is the same as [Armstrong, Definition (2.1)]. Note that [Sutherland, Definition 7.1] insists on X being non-empty, but we do not require this.

The **De Morgan laws,** Lemma 1.3: see for example [Willard, Theorem 1.4].

The **antidiscrete topology** is called **indiscrete** in [Armstrong, Problem 29] and [Sutherland, Example 7.5]. The proof that **cofinite topology** is a topology in Proposition 1.4 solves [Sutherland, Exercise 7.5].