MATH11112 Real Analysis (2025). Exercise sheet for week 05 Interval of convergence of a power series. The functions e^x, $\ln x$ — SOLUTIONS

Homework

Q18. (i) Find the radius of convergence R of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

(ii) Find I, the interval of convergence of the power series given in (i). Determine the behaviour of the power series at the endpoints of I: that is, determine, justifying your claims, whether

- the series is divergent, conditionally convergent, or absolutely convergent at $x = -R$;
- the series is divergent, conditionally convergent, or absolutely convergent at $x = R$.

(iii) If $x \in I$, denote by $f(x)$ the sum of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Write $f(x) - f(0)$ in the form $F_0(x)(x - 0)$ where the function $F_0 : I \to \mathbb{R}$ is continuous. Give a reason why F_0 is continuous (quote a result from the course). Conclude that f is differentiable at $x = 0$ and find $f'(0)$.

Q18. Solution. (i) *Students should learn the following important method of finding the radius of convergence of a power series. This method is described, and illustrated by two examples, in Chapter 3 of the lecture notes. This question gives us yet another example of the method.*

To find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, we **test absolute convergence** by applying the Ratio Test to $\sum_{n=1}^{\infty} \left|\frac{x^n}{n^2}\right|$. Note: we are testing absolute convergence, and so we use the absolute value in $\left|\frac{x^n}{n^2}\right|$.

When $x = 0$, this series is convergent, and when $x \neq 0$, we have

$$\ell = \lim_{n\to\infty} \frac{|x^{n+1}/(n+1)^2|}{|x^n/n^2|} = \lim_{n\to\infty} \frac{n^2|x|}{(n+1)^2} = \lim_{n\to\infty} \frac{1^2|x|}{(1+1/n)^2} = \frac{1^2|x|}{(1+0)^2} = |x|.$$

The Ratio Test tells us that the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is absolutely convergent when $\ell = |x| < 1$, and is not absolutely convergent when $\ell = |x| > 1$.

As we know from the course (see Corollary 3.2), the radius of convergence is the unique R such that the power series is absolutely convergent for $|x| < R$ and is not absolutely convergent (in fact, divergent) when $|x| > R$. Hence for the given series **the radius of convergence is $R = 1$.**

(ii) By part (i), $R = 1$, so by a result from the course, Corollary 3.2, the endpoints of I are -1 and 1. It remains to determine convergence at $x = -1$ and at $x = 1$.

At $x = 1$ we obtain the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We know from earlier work, see Q3, that this series is convergent. Its terms are non-negative hence **the series is absolutely convergent when $x = 1$.**

Substituting $x = -1$, we obtain the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. The series of absolute values is $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent as we noted above. Thus, **the series is absolutely convergent when $x = -1$.**

By definition, the interval of convergence I is the set of all points x where the power series is convergent. We have shown that the given power series is convergent at the endpoints -1 and 1 of I, hence both endpoints are contained in I, making I a closed interval. We conclude that I is the **closed interval** $[-1, 1]$.

(iii) We note that $f(0) = 0$, so we need to write $f(x)$ in the form $F_0(x)x$. We claim that

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \Big(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2}\Big)x$$

which students, sadly, tend to write down without any justification. This is true by Algebra of Infinite Sums (AoIS), **assuming** that at least one of the series involved is convergent.

We put $F_0(x)$ to be the sum of the series $\sum_{n=1}^{\infty} \dfrac{x^{n-1}}{n^2}$ which is absolutely convergent when $x \in [-1, 1]$. We quote the following result from the course:

> Theorem 3.5 (*a function defined by a power series is continuous*). Let $C(x)$ denote the sum of a power series with radius of convergence R. Then $C(x)$ is a continuous function on the interval $(-R, R)$.

By the Theorem, $F_0(x)$ is continuous on the interval $(-1, 1)$. So in particular, since $0 \in (-1, 1)$, the function $F_0(x)$ is continuous at $x = 0$.

By a result from the course, Proposition 4.6 (differentiability at a means continuity of the slope function at a), continuity of $F_0(x)$ implies that $f(x)$ is differentiable at 0 with $f'(0) = F_0(0)$. It is not necessary to invoke the Proposition in this case, as we can use the definition of derivative: $\frac{f(x)-f(0)}{x-0} = F_0(x)$ if $x \neq 0$, so $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} F_0(x) = F_0(0)$ by continuity of F_0 at 0. Since $F_0(x) = 1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \dots$, we have $f'(0) = F_0(0) = 1.$

Supervision work

Q19. Exercise on the Nullity Test, Absolute Convergence Theorem, Alternating Series Test. The three series given below contain both positive and negative terms. For each series, decide whether it is absolutely convergent, conditionally convergent or divergent, giving reasons.

(i) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{\sqrt{n}}$ (ii) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}(n+1)}{n}$ (iii) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}d_n}{n^2}$, where $d_n \in \{1, 2, \dots, 9\}$ is the first decimal digit of n (e.g., $d_5 = d_{567} = 5$). Is the Alternating Series Test applicable in (iii)?

Q19. Solution. (i) The given series is

- **not absolutely convergent:** $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots$ is a divergent series, seen on example sheet 1;
- convergent by the **Alternating Series Test,** as $\frac{1}{\sqrt{n}}$ is a decreasing sequence with limit 0;
- hence **conditionally convergent.**

(ii) **Divergent by the Nullity Test:** $\frac{(-1)^{n+1}(n+1)}{n}$ does not have limit 0 as $n \to \infty$.

(Alternating Series Test is not applicable: $(\frac{n+1}{n})$, though decreasing, does not have limit 0.)

(iii) The series of absolute values is $\sum_{n=1}^{\infty} |\frac{(-1)^{n+1}d_n}{n^2}| = \sum_{n=1}^{\infty} \frac{d_n}{n^2}$. We have $0 \le \frac{d_n}{n^2} \le \frac{9}{n^2}$, hence the series $\sum_{n=1}^{\infty} \frac{d_n}{n^2}$ is convergent by comparison with the convergent series $\sum_{n=1}^{\infty} \frac{9}{n^2}$. This means that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}d_n}{n^2}$ is **absolutely convergent.**

Note that the Alternating Series Test is not applicable, because the sequence $(\frac{d_n}{n^2})$ is not a decreasing sequence: $\frac{d_{19}}{19^2} = \frac{1}{361}$ yet $\frac{d_{20}}{20^2} = \frac{1}{200} > \frac{1}{361}$.

Q20. e^x **grows faster than** x^m**.** In MFA we saw that "exponential beats polynomial" for sequences; here is a version for functions. For positive functions f, g defined on $(1, +\infty)$ with $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = +\infty$, we say that $f(x)$ **grows faster** than $g(x)$ if $\lim_{x\to+\infty} \frac{g(x)}{f(x)} = 0$. Use the power series definition of e^x to show that for **every** $m \in \mathbb{N}$ (no matter how large), e^x grows faster than x^m as $x \to +\infty$.

Q20. Solution. The series $e^x = 1 + x + \dfrac{x^2}{2!} + \dots + \dfrac{x^m}{m!} + \dots$ contains the term $\dfrac{x^m}{m!}$, and all terms are positive since we consider positive x, so

$$e^x > \frac{x^m}{m!} \quad \text{when } x > 0.$$

Yet this isn't quite sufficient for showing that e^x "grows faster". Rather, we use

$$e^x > \frac{x^{m+1}}{(m+1)!}$$

to conclude that $0 < \dfrac{x^m}{e^x} \le \dfrac{x^m}{x^{m+1}/(m+1)!} = \dfrac{(m+1)!}{x}$. Since $\lim_{x\to\infty} \dfrac{(m+1)!}{x} = 0$ (it is x which tends to infinity, and $(m+1)!$ stays constant), by Sandwich Rule $\lim_{x\to\infty} \dfrac{x^m}{e^x} = 0$.

Thus, e^x grows faster than x^m.

Q21. The inequalities $e^x \ge 1 + x$ **and** $\ln(1+y) \le y$**.** (a) Prove that $e^x \ge 1 + x$ when $x \ge 0$ using the power series definition of e^x. Prove the same for $x \le -1$ using positivity of e^x.

(b) If $-1 < x < 0$, put $t = -x$ and show that $e^t \le \frac{1}{1-t}$ by comparing the power series expansions for e^t and $\frac{1}{1-t}$. What inequality between e^x and $1 + x$ does this give?

(c) Deduce the inequality $\ln(1+y) \le y$ by applying $\ln$ to both sides of $e^x \ge 1 + x$. Which properties of the $\ln$ function are we using here? For which $y \in \mathbb{R}$ does $\ln(1+y) \le y$ hold?

Alternatively, these inequalities can be proved using the Mean Value Theorem, to be seen in week 6.

Q21. Solution. (a) If $x \geq 0$, $e^x - (1+x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ is the sum of a series with non-negative terms. It must be non-negative, so $e^x - (1+x) \geq 0$. If $x \leq -1$ then $1 + x \leq 0 < e^x$.

(b) $e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \dots$ by definition; $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$ when $|t| < 1$ by the geometric sum formula. Since $0 \leq \frac{t^n}{n!} \leq t^n$ for positive t, by Comparison Test $e^t \leq \frac{1}{1-t}$ for $0 < t < 1$.

This means that when $-1 < x < 0$, $e^{-x} = \frac{1}{e^x} \leq \frac{1}{1+x}$. Both sides are positive, so $e^x \geq 1 + x$.

(c) Since $\ln$ is an increasing function defined on $(0, +\infty)$, the inequality $e^y \geq 1 + y$ implies $\ln e^y = \ln(1 + y)$ if both e^y and $1 + y$ are in $(0, +\infty)$. This holds when $y > -1$. Since $\ln$ is inverse to e^y, we have $y \geq \ln(1 + y)$ whenever $y > -1$.

Q22. A function differentiable at just one point. As discussed in the lectures, the function $|x|$ is differentiable everywhere **except** the point $x = 0$. Our next example achieves the opposite.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be given by $h(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$ Put $f(x) = x^2 h(x)$.

(i) Show that $f(x)$ is differentiable at the point $x = 0$.

(ii) Show that $f(x)$ is **not** differentiable at all other points except 0. (*Hint*: is f continuous?)

Q22. Solution. (i) $f(0) = 0$ so "differentiable" means that $\lim_{x\to 0} \frac{f(x)-0}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ exists.

When calculating $\lim_{x\to 0}$, we will assume (as we may) that $x \neq 0$. Then $\frac{f(x)}{x} = \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$ We therefore have $|\frac{f(x)}{x}| \leq |x|$, or, opening out,

$$-|x| \leq f(x)/x \leq |x|.$$

As $-|x|, |x|$ have limit 0 when $x \to 0$, by Sandwich Rule $\lim_{x\to 0} \frac{f(x)}{x}$ exists (and equals 0), q.e.d.

(ii) By a result from this course, Theorem 4.4, "f is differentiable at a" implies "f is continuous at a". We will show that f is not continuous (hence not differentiable) at any $a \neq 0$.

Given any $a \in \mathbb{R}$, there exist sequences (x_n), (y_n) where x_n is rational and y_n is irrational for all n, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$. (*This is easy to achieve, for example by taking x_n to be the decimal expansion of a truncated after n digits, and putting $y_n = x_n - \frac{\sqrt{2}}{n}$.*)

We have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 = a^2$ by Algebra of Limits, and $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 0 = 0$. The two limits are different if $a \neq 0$, yet by a result from MFA (Proposition 11.4.1, a criterion for continuity), for a continuous function all such limits must be $f(a)$. Hence f is discontinuous at $a \neq 0$.

Extra exercises

Attempt these questions in your own time and compare your answers with the model solutions published on Monday in week 6. *Note that there is no examples class in week 6 due to Online Coursework Test.*

Q23. Any positive power of x grows faster than $\ln x$. Deduce formally from Q20 that for every positive β (no matter how small), x^β grows faster than $\ln x$.

Q23. Solution. Write $\frac{\ln x}{x^\beta}$ as $\frac{1}{\beta}\frac{\ln(x^\beta)}{x^\beta}$. If $y = \ln x^\beta$, then by Q20 for each $\varepsilon > 0$ there exists $K > 0$ such that $y > K \Rightarrow |\frac{y}{e^y}| < \varepsilon\beta$. Equivalently, $x > e^{K/\beta} \Rightarrow |\frac{\ln x}{x^\beta}| < \varepsilon$. We have verified the definition of "$\lim_{x\to+\infty} \frac{\ln x}{x^\beta} = 0$".

Q24. Divergence of $\sum \frac{1}{n \ln n}$. Since $\ln n \geq 1$ if $n \geq 3$ (*justify this!*), we have $\frac{1}{n\ln n} \leq \frac{1}{n}$. Q23 gives, for any fixed $\varepsilon > 0$, a lower bound: there is $c > 0$ such that, for all $n \geq 3$,

$$\frac{c}{n^{1+\varepsilon}} \leq \frac{1}{n \ln n} \leq \frac{1}{n}.$$

Yet, as $\sum \frac{1}{n^{1+\varepsilon}} < +\infty$ and $\sum \frac{1}{n} = +\infty$, this tells us nothing about convergence of $\sum \frac{1}{n\ln n}$.

(i) Use the Cauchy Condensation Test from Q7 to show that $\sum \frac{1}{n \ln n}$ is **divergent.**

(ii) Likewise, investigate the convergence of $\sum \frac{1}{n(\ln n)^2}$ and $\sum \frac{1}{n \ln n \ln(\ln n)}$.

Q24. Solution. *The Cauchy Condensation Test is only seen in extra exercises; it is not necessary to learn this test for the exam.* Recall, the Cauchy Condensation Test says that if $a_1 \geq a_2 \geq \ldots$ are non-negative, then $a_1 + a_2 + a_3 + \ldots$ is convergent if, and only if, the condensed series $\sum_{k=0}^\infty 2^k a_{2^k}$ is convergent.

The sequence $(n \ln n)_{n\geq 1}$ is increasing as both n and $\ln n$ are non-negative and increasing. Moreover, $\ln n > 0$ for $n \geq 2$, and the bound $e \leq 3$ from Q5 gives $\ln n \geq \ln e = 1$ for all $n \geq 3$.

Therefore, the test can be applied to the series $\sum_{n\geq 2} \frac{1}{n\ln n}$. The condensed series is

$$\sum_{k\geq 1} 2^k \frac{1}{2^k \ln(2^k)} = \sum_{k\geq 1} \frac{1}{k \ln 2}.$$

Up to the factor $\frac{1}{\ln 2} > 0$, this is the harmonic series, which is divergent. Hence $\sum \frac{1}{n \ln n} = +\infty$.

(ii) Similarly, the condensed series of $\sum \frac{1}{n(\ln n)^2}$ is $\sum 2^k \frac{1}{2^k(k\ln 2)^2} = \sum \frac{1/\ln 2}{k^2} < +\infty$ (the sum of this series is $\frac{\pi^2}{6 \ln 2}$) which by the Cauchy Condensation Test means $\sum \frac{1}{n(\ln n)^2} < +\infty$.

Yet the condensed series of $\sum \frac{1}{n(\ln n)(\ln \ln n)}$ is

$$\sum 2^k \frac{1}{2^k (k\ln 2)(\ln(k \ln 2))} = \sum \frac{1/\ln 2}{k(\ln k + \ln\ln 2)}.$$

Careful, we should number the series $\sum \frac{1}{n(\ln n)(\ln\ln n)}$ from $n = 3$ to avoid negative terms, and the condensed series starts at $k = 2$. Since $\ln \ln 2 < 0$, we have $\frac{1}{k(\ln k + \ln \ln 2)} \geq \frac{1}{k \ln k}$. The latter series diverges to $+\infty$ by part (i), hence by the Cauchy Condensation Test $\sum \frac{1}{n(\ln n)(\ln \ln n)} = +\infty$.

Q25. $\dfrac{1}{k+1} \leq \ln\Big(1 + \dfrac{1}{k}\Big) \leq \dfrac{1}{k}$ for all $k \in \mathbb{N}$:
deduce this from the inequality $\ln(1+y) \leq y$ for all $y > -1$ (Q21). *Hint*: the "$\leq \frac{1}{k}$" bound is obvious if you put $y = \frac{1}{k}$; think of a *negative* y which will give you $\frac{1}{k+1} \leq \ln(1+\frac{1}{k})$.

Q25. Solution. Putting $y = \frac{1}{k}$ in the inequality $\ln(1+y) \leq y$ from Q25 gives $\ln(1+\frac{1}{k}) \leq \frac{1}{k}$. Putting $y = -\frac{1}{k+1}$, allowable as $-\frac{1}{k+1} > -1$, gives the required lower bound:

$$\ln \tfrac{k}{k+1} \leq -\tfrac{1}{k+1} \quad \Leftrightarrow \quad -\ln \tfrac{k}{k+1} = \ln \tfrac{k+1}{k} = \ln(1+\tfrac{1}{k}) \geq \tfrac{1}{k+1}.$$

Q26. Partial sums of the harmonic series have logarithmic growth.
(a) Show that $\Big(1+\dfrac{1}{1}\Big)\Big(1+\dfrac{1}{2}\Big)\Big(1+\dfrac{1}{3}\Big)\dots\Big(1+\dfrac{1}{n-1}\Big) = n.$
(b) Apply the double bound for $\ln(1+\frac{1}{k})$ from Q25 to the result of (a) to deduce that

$$\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \leq \ln n \leq 1 + \frac{1}{2} + \dots + \frac{1}{n-1}.$$

Alternatively, these inequalities follow by considering lower and upper Riemann sums when integrating $\frac{1}{x}$ *over* $[1, n]$ *— revisit this exercise at the end of the course.*

(c) If s_n is the n^{th} partial sum of the harmonic series, show that $\lim\limits_{n\to\infty} \dfrac{s_n}{\ln n} = 1.$

Comment: one says that $s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ **grow logarithmically,** which is slower than n^{ε} for any positive ε. Informally, we have the order

logarithmic growth $<$ polynomial growth $<$ exponential growth.

Q26. Solution. (a) The product, $\frac{2}{1} \times \frac{3}{2} \times \dots \times \frac{n}{n-1}$, clearly becomes $\frac{n}{1} = n$ after cancellation.

(b) Applying the increasing function $\ln$ to both sides of the equation (a) and using the bounds $\frac{1}{k+1} \leq \ln(1+\frac{1}{k}) \leq \frac{1}{k}$ for all $k = 1, 2, \dots, n-1$ gives the required inequality.

(c) By (b), $\ln n \leq s_n \leq \ln n + 1$. Dividing through by $\ln n$, we obtain $1 \leq \frac{s_n}{\ln n} \leq 1 + \frac{1}{\ln n}$ whence $\frac{s_n}{\ln n} \to 1$ by Sandwich Rule as $\frac{1}{\ln n} \to 0$.

Q27. Differentiate x^x. The function x^x is formally defined for all positive x as $e^{x \ln x}$. Use the rules of differentiation and the result that $\frac{d}{dx} \ln x = \frac{1}{x}$ to find the derivative $\frac{d}{dx}(x^x)$.

Q27. Solution. $\frac{d}{dx}(x \ln x) = x' \ln x + x \ln' x = \ln x + 1$ by Product Rule, so $\frac{d}{dx}(x^x) = \frac{d}{dx} \exp(x \ln x) = \exp'(x \ln x)(\ln x + 1)$. Since $\exp' = \exp$, we have $\frac{d}{dx}(x^x) = x^x(\ln x + 1)$.

Q28. Use of $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$ to estimate probability – e.g., in telecommunications.

This result, (soon to be) proved in the lectures, is useful for estimating the probability of a near-certain event occurring N times in a row, where N is very large. Let us consider an example.

In the 20^{th} century digital data was often transferred over analogue telephone lines, via modems. Suppose you download 10 kbytes (approx. 80000 bits) of software via a modem, and each bit has probability of 0.999 (realistic!) to be transmitted without an error. If one or more errors occur, the software won't run and you need to redownload. Estimate the probability that no errors occur.

The answer is 0.999^{80000} but it is not clear how large or small is this number. Use $\lim_{n\to\infty}(1+\frac{x}{n})^n$ to show that the probability of no errors occurring is, in fact, **very** small.

Comment: In reality, parity check was applied to small groups of bits which detected most errors and necessitated retransmission of only small amounts of data. In modern networks, the probability of no error occurring in a single bit being transmitted is more like $1-10^{-12}$. Still, a large file will have an exponentially small probability of arriving without errors, which means that more sophisticated error detection algorithms have to be used.

Q28. Solution. $0.999^{80000} = (1-\frac{1}{1000})^{80000} = (1+\frac{-80}{80000})^{80000} \approx e^{-80}$, using $x=-80$ and $n=80000$.

Now, e^{-80} is vanishingly small. For a rough estimate, use the fact that $e^3 > 20$. Then $e^{-80} < 20^{-26}e^{-2} = 2^{-20} \times \frac{1}{64e^2} \times 10^{-26}$.

Here, 2^{-20} is less than $\frac{1}{1000000}$, and $e^2 > 7$ so $64e^2 > 400$.

The result is then less than $\frac{1}{4\times 10^8}10^{-26} = 2.5\times 10^{-35}$.

Compare this with a more accurate value for e^{-80}, and with a more accurate value for 0.999^{80000}; observe how good is the approximation of 0.999^{80000} by e^{-80} (exercise).

In any case, note that 10^{35} is greater than, say, the number of oxygen atoms in a room, or the total number of all bacteria on Earth.