MATH11112 Real Analysis (2025). Exercise sheet for week 05
Interval of convergence of a power series. The functions e^x, $\ln x$

Homework

Attempt all parts of Q18 and submit your work online before 2pm on Tuesday 25th February. You will find the Gradescope submission link in the Week 5 folder on Blackboard.

Q18. (i) Find the radius of convergence R of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

(ii) Find I, the interval of convergence of the power series given in (i). Determine the behaviour of the power series at the endpoints of I: that is, determine, justifying your claims, whether

- the series is divergent, conditionally convergent, or absolutely convergent at $x = -R$;
- the series is divergent, conditionally convergent, or absolutely convergent at $x = R$.

(iii) If $x \in I$, denote by $f(x)$ the sum of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Write $f(x) - f(0)$ in the form $F_0(x)(x-0)$ where the function $F_0 \colon I \to \mathbb{R}$ is continuous. Give a reason why F_0 is continuous (quote a result from the course). Conclude that f is differentiable at $x = 0$ and find $f'(0)$.

Supervision work

Attempt these questions and bring your solutions to the Week 5 supervision class for discussion.

Q19. Exercise on the Nullity Test, Absolute Convergence Theorem, Alternating Series Test. The three series given below contain both positive and negative terms. For each series, decide whether it is absolutely convergent, conditionally convergent or divergent, giving reasons.

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$ (iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} d_n}{n^2}$, where $d_n \in \{1, 2, \dots, 9\}$ is the first decimal digit of n (e.g., $d_5 = d_{567} = 5$). Is the Alternating Series Test applicable in (iii)?

Q20. e^x grows faster than x^m. In MFA we saw that "exponential beats polynomial" for sequences; here is a version for functions. For positive functions f, g defined on $(1, +\infty)$ with $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = +\infty$, we say that $f(x)$ **grows faster** than $g(x)$ if $\lim_{x\to+\infty} \frac{g(x)}{f(x)} = 0$.

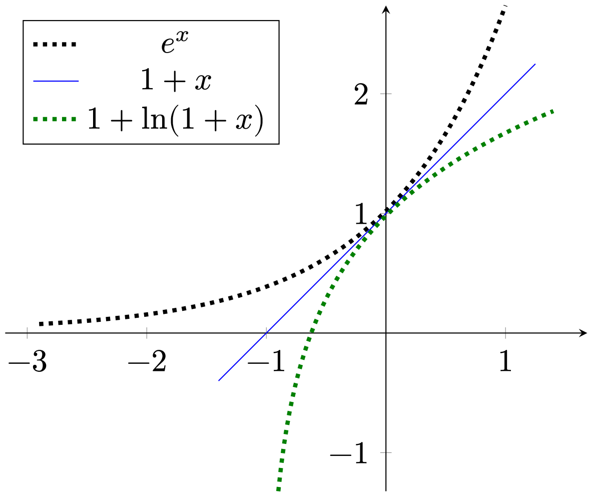
Use the power series definition of e^x to show that for **every** $m \in \mathbb{N}$ (no matter how large), e^x grows faster than x^m as $x \to +\infty$.

Q21. The inequalities $e^x \ge 1 + x$ and $\ln(1 + y) \le y$. (a) Prove that $e^x \ge 1 + x$ when $x \ge 0$ using the power series definition of e^x. Prove the same for $x \le -1$ using positivity of e^x.

(b) If $-1 < x < 0$, put $t = -x$ and show that $e^t \leq \frac{1}{1-t}$ by comparing the power series expansions for e^t and $\frac{1}{1-t}$. What inequality between e^x and $1 + x$ does this give?

(c) Deduce the inequality $\ln(1 + y) \leq y$ by applying $\ln$ to both sides of $e^x \geq 1 + x$. Which properties of the $\ln$ function are we using here? For which $y \in \mathbb{R}$ does $\ln(1 + y) \leq y$ hold?

Alternatively, these inequalities can be proved using the Mean Value Theorem, to be seen in week 6.



The graph of e^x is above the line $y = 1 + x$ which is above the graph of $y = 1 + \ln(1 + x)$.

Q22. A function differentiable at just one point. As discussed in the lectures, the function $|x|$ is differentiable everywhere **except** the point $x = 0$. Our next example achieves the opposite.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be given by $h(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$ Put $f(x) = x^2 h(x)$.

(i) Show that $f(x)$ is differentiable at the point $x = 0$.

(ii) Show that $f(x)$ is **not** differentiable at all other points except 0. (*Hint*: is f continuous?)

Extra exercises

Attempt these questions in your own time and compare your answers with the model solutions published on Monday in week 6. *Note that there is no examples class in week 6 due to Online Coursework Test.*

Q23. Any positive power of x grows faster than $\ln x$. Deduce formally from Q20 that for every positive β (no matter how small), x^β grows faster than $\ln x$.

Q24. Divergence of $\sum \frac{1}{n \ln n}$. Since $\ln n \geq 1$ if $n \geq 3$ (*justify this!*), we have $\frac{1}{n \ln n} \leq \frac{1}{n}$. Q23 gives, for any fixed $\varepsilon > 0$, a lower bound: there is $c > 0$ such that, for all $n \geq 3$,

$$\frac{c}{n^{1+\varepsilon}} \leq \frac{1}{n \ln n} \leq \frac{1}{n}.$$

Yet, as $\sum \frac{1}{n^{1+\varepsilon}} < +\infty$ and $\sum \frac{1}{n} = +\infty$, this tells us nothing about convergence of $\sum \frac{1}{n \ln n}$.

(i) Use the Cauchy Condensation Test from Q7 to show that $\sum \frac{1}{n \ln n}$ is **divergent.**

(ii) Likewise, investigate the convergence of $\sum \frac{1}{n(\ln n)^2}$ and $\sum \frac{1}{n \ln n \ln(\ln n)}$.

Q25. $\dfrac{1}{k+1} \le \ln\Big(1+\dfrac{1}{k}\Big) \le \dfrac{1}{k}$ for all $k \in \mathbb{N}$:
deduce this from the inequality $\ln(1+y) \le y$ for all $y > -1$ (Q21). *Hint*: the "$\le \frac{1}{k}$" bound is obvious if you put $y = \frac{1}{k}$; think of a *negative* y which will give you $\frac{1}{k+1} \le \ln(1+\frac{1}{k})$.

Q26. Partial sums of the harmonic series have logarithmic growth.

(a) Show that $\Big(1+\dfrac{1}{1}\Big)\Big(1+\dfrac{1}{2}\Big)\Big(1+\dfrac{1}{3}\Big)\dots\Big(1+\dfrac{1}{n-1}\Big) = n.$

(b) Apply the double bound for $\ln(1+\frac{1}{k})$ from Q25 to the result of (a) to deduce that

$$\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} \le \ln n \le 1+\frac{1}{2}+\dots+\frac{1}{n-1}.$$

Alternatively, these inequalities follow by considering lower and upper Riemann sums when integrating $\frac{1}{x}$ *over* $[1,n]$ *— revisit this exercise at the end of the course.*

(c) If s_n is the n^{th} partial sum of the harmonic series, show that $\lim\limits_{n\to\infty} \dfrac{s_n}{\ln n} = 1.$

Comment: one says that $s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ **grow logarithmically,** which is slower than n^{ε} for any positive ε. Informally, we have the order

logarithmic growth $<$ polynomial growth $<$ exponential growth.

Q27. Differentiate x^x. The function x^x is formally defined for all positive x as $e^{x \ln x}$. Use the rules of differentiation and the result that $\frac{d}{dx}\ln x = \frac{1}{x}$ to find the derivative $\frac{d}{dx}(x^x)$.

Q28. Use of $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$ to estimate probability – e.g., in telecommunications.

This result, (soon to be) proved in the lectures, is useful for estimating the probability of a near-certain event occurring N times in a row, where N is very large. Let us consider an example.

In the 20^{th} century digital data was often transferred over analogue telephone lines, via modems. Suppose you download 10 kbytes (approx. 80000 bits) of software via a modem, and each bit has probability of 0.999 (realistic!) to be transmitted without an error. If one or more errors occur, the software won't run and you need to redownload. Estimate the probability that no errors occur.

The answer is 0.999^{80000} but it is not clear how large or small is this number. Use $\lim_{n\to\infty}(1+\frac{x}{n})^n$ to show that the probability of no errors occurring is, in fact, **very** small.

Comment: In reality, parity check was applied to small groups of bits which detected most errors and necessitated retransmission of only small amounts of data. In modern networks, the probability of no error occurring in a single bit being transmitted is more like $1-10^{-12}$. Still, a large file will have an exponentially small probability of arriving without errors, which means that more sophisticated error detection algorithms have to be used.