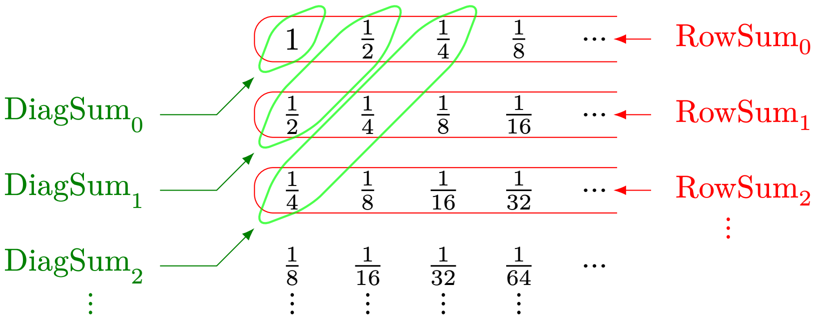
MATH11112 Real Analysis (2025). Exercise sheet for week 03 Convergence tests — SOLUTIONS

Homework

Q9. Consider the double series with general term given by $a_{m,n} = \dfrac{1}{2^{m+n}}$ for $m, n \geq 0$. Let $\color{red}{\mathrm{RowSum}_m}$ denote the sum $\sum_{n=0}^{\infty} a_{m,n}$ of the entries in the mth row, and $\color{green}{\mathrm{DiagSum}_d}$ denote the sum $\sum\limits_{m+n=d} a_{m,n}$ of the entries on the dth diagonal:



(i) Write down a formula expressing RowSum_m in terms of m. (*Hint:* each row of the given double series is a geometric series.)

(ii) Let $S = \sum_{m=0}^{\infty} \mathrm{RowSum}_m$; find the numerical value of S.

(iii) Write down a formula expressing $\mathrm{DiagSum}_d$ in terms of d.

(iv) Since the given double series has non-negative terms, a result from the lectures says that $\sum_{d=0}^{\infty} \mathrm{DiagSum}_d = S$. Use this result to calculate $\sum\limits_{n=0}^{\infty} \dfrac{n}{2^n}$, briefly explaining what you do.

Q9. Solution. (i) RowSum_m is the sum of the geometric series $\dfrac{1}{2^m} + \dfrac{1}{2^{m+1}} + \dfrac{1}{2^{m+2}} + \ldots$ with initial term $\dfrac{1}{2^m}$ and ratio $\dfrac{1}{2}$. The formula for the sum of geometric series gives $\dfrac{1/2^m}{1-1/2}$. Hence $\boxed{\mathrm{RowSum}_m = \dfrac{1}{2^{m-1}}.}$

(ii) $S = \sum\limits_{m=0}^{\infty} \dfrac{1}{2^{m-1}}$, this is the sum of geometric series $2 + 1 + \dfrac{1}{2} + \cdots = \dfrac{2}{1-1/2}$. Hence $\boxed{S = 4.}$

(iii) The dth diagonal contains $d+1$ entries, each equal to $\dfrac{1}{2^d}$. Hence $\boxed{\mathrm{DiagSum}_d = \dfrac{d+1}{2^d}.}$

(iv) The result quoted in the question says that numbers in a double series can be summed by diagonals or by rows — the answer will be the same (if all the numbers are non-negative). Thus, $\sum\limits_{d=0}^{\infty} \dfrac{d+1}{2^d} = 4$. Using Algebra of Infinite Sums, part (ii), and the sum of geometric series, we calculate

$$\sum_{n=0}^{\infty} \frac{n}{2^n} = \sum_{n=0}^{\infty} \frac{n+1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{2^n} = 4 - 2 = \boxed{2.}$$

Supervision work

Q10. Exercise on the Ratio Test. The **Fibonacci sequence** $F_1, F_2, F_3, \ldots$ is defined by the rule $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_{n-1} + F_n$ for all $n \geq 2$. Consider the **Fibonacci reciprocal series**

$$\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{F_n}.$$

(i) Write down the first six terms of the series.

(ii) Write down the explicit formula for F_n in terms of the two roots, α and β, of the quadratic equation $x^2 = x + 1$. This formula should be known to you from Section 2 of MFA (Semester 1).

(iii) Apply the Ratio Test to the series $\sum\limits_{n=1}^{\infty} \frac{1}{F_n}$. Use the formula from (ii) to find the limit ℓ needed for the test. Then, based on the numerical value of ℓ, determine whether the series is convergent.

Q10. Solution. (i) $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \ldots$

(ii) It was proved in MFA, using mathematical induction, that the n^{th} term of the Fibonacci sequence $F_1, F_2, F_3, \ldots$ is given by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \qquad \text{where } \alpha \text{ is the } \textbf{golden ratio } \frac{1+\sqrt{5}}{2}, \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

In other words, α is the positive root, and β is the negative root of the quadratic equation $x^2 = x + 1$. It is also useful to note the approximate values $\alpha = 1.618\ldots$, $\beta = 1 - \alpha = -0.618\ldots$ Both the Fibonacci sequence and the golden ratio occur in nature as well as in the arts, and are widely discussed in popular mathematical literarure.

(iii) We apply the Ratio Test to the series $\sum\limits_{n=1}^{\infty} \frac{1}{F_n}$ which has positive terms. Calculate the limit

$$\ell = \lim_{n\to\infty} \frac{1/F_{n+1}}{1/F_n} = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lim_{n\to\infty} \frac{(\alpha^n - \beta^n)/\sqrt{5}}{(\alpha^{n+1} - \beta^{n+1})/\sqrt{5}} = \lim_{n\to\infty} \frac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}}$$
$$= \lim_{n\to\infty} \frac{1 - (\frac{\beta}{\alpha})^n}{\alpha - \beta(\frac{\beta}{\alpha})^n}.$$

Since $|\frac{\beta}{\alpha}| < 1$ (important: do not forget the modulus, $\frac{\beta}{\alpha} < 1$ says nothing as $\frac{\beta}{\alpha}$ is negative), by a result from MFA $(\frac{\beta}{\alpha})^n \to 0$ as $n \to \infty$. We now use Algebra of Limits to calculate

$$\ell = \frac{1 - 0}{\alpha - \beta \cdot 0} = \alpha^{-1} \quad (= \alpha - 1 \approx 0.618).$$

Since $0 \leq \ell < 1$, the series $\sum\limits_{n=1}^{\infty} \frac{1}{F_n}$ is convergent by the Ratio Test.

Q11. Exercises on the Comparison Test. (i) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent, by comparing it with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(ii) Show that $\sum_{n=1}^{\infty} \frac{100}{n^2 - 0.5\sqrt{n}}$ is convergent, by comparing it with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, multiplied by a suitable constant.

(iii) Are $\sum_{n=1}^{\infty} \frac{1}{n-0.5}$ and $\sum_{n=1}^{\infty} \frac{1}{n+100.5}$ convergent or divergent?

Q11. Solution. (i) Since $n^2+1 \geq n^2$, we have $0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (see Q3), $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is also convergent, by a straightforward application of the Comparison Test.

(ii) Careful, $\frac{100}{n^2 - 0.5\sqrt{n}}$ is not bounded above by $100 \times \frac{1}{n^2}$ — we need an extra step here. Note that $\sqrt{n} \leq n \leq n^2$ for all $n \geq 1$, and so

$$n^2 - 0.5\sqrt{n} \geq n^2 - 0.5n^2 = 0.5n^2 \quad \Rightarrow \quad \frac{100}{n^2 - 0.5\sqrt{n}} \leq \frac{100}{0.5n^2} = \frac{200}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{200}{n^2}$ is convergent by Algebra of Infinite Sums, because it is obtained by multiplying a convergent series, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, by a constant, 200. Hence $\sum_{n=1}^{\infty} \frac{100}{n^2 - 0.5\sqrt{n}}$ is convergent by the Comparison Test.

(iii) Note that to prove divergence, the Comparison Test can be used in the contrapositive: $0 \leq a_n \leq b_n$ for all n, $\sum_{n=1}^{\infty} a_n$ is divergent $\Rightarrow \sum_{n=1}^{\infty} b_n$ is divergent. That is, a given series must be **bounded below** by a series known to be divergent.

Observe that $n \geq n - 0.5$ and so $\frac{1}{n} \leq \frac{1}{n-0.5}$ for all $n \in \mathbb{N}$. We know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, so by the (contrapositive of the) Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n-0.5}$ is divergent.

This approach does not work verbatim for the series $\sum_{n=1}^{\infty} \frac{1}{n+100.5}$ because $\frac{1}{n+100.5}$ is not bounded below by $\frac{1}{n}$. Here are two of the possible ways to finish the proof that $\sum_{n=1}^{\infty} \frac{1}{n+100.5}$ is divergent.

Way 1. Note that $100.5 \leq 100.5n$ for all $n \geq 1$, and so $\frac{1}{n+100.5} \geq \frac{1}{101.5n}$. The series $\sum_{n=1}^{\infty} \frac{1}{101.5n}$ is divergent (because if it were convergent, multiplying it by 101.5 would result, by AoIS, in a convergent series $\sum \frac{1}{n}$ which is absurd). Hence by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n+100.5}$ is divergent.

Way 2. Note that $\frac{1}{n+100.5} \geq \frac{1}{n+101}$, so the partial sums $s_n = \frac{1}{1+100.5} + \frac{1}{2+100.5} + \cdots + \frac{1}{n+100.5}$ are bounded below by $h_{n+101} - h_{101}$ where h_n are partial sums of the harmonic series. Since $h_n \to +\infty$ as $n \to \infty$, we have $s_n \to +\infty$, and so $\sum \frac{1}{n+100.5}$ is divergent.

Q12. (a question from a past exam paper) Use Algebra of Infinite Sums to prove that the following series is convergent: $\sum_{n=1}^{\infty}\left(\frac{1000}{n^2}+\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$.

Q12. Solution. Put $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$.

By the result of Q3, the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

In the sum $b_1 + \dots + b_n = (\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}) + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}) + \dots + (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})$, intermediate terms cancel to give $1 - \frac{1}{\sqrt{n+1}}$. (This is a "telescoping sum" similar to Q2.) We use $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ and Algebra of Limits to conclude that, as $n \to \infty$, the partial sums $b_1 + \dots + b_n$ have limit $1 - 0 = 1$. By definition, the series $\sum\limits_{n=1}^{\infty} b_n$ is convergent, with sum 1.

The given series is $\sum\limits_{n=1}^{\infty}(1000a_n + b_n)$. By Algebra of Infinite Sums, it is convergent.

Q13. Absolute vs. conditional convergence. (i) Show: $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots$ is a convergent series with sum 0.

(ii) Determine whether the series in (i) is **absolutely convergent** or **conditionally convergent.** (By definition, "conditionally convergent" means convergent but not absolutely convergent.)

Q13. Solution. (i) The partial sums of the given series are $s_1 = 1$, $s_2 = 0$, $s_3 = \frac{1}{2}$, $s_4 = 0$, $s_5 = \frac{1}{3}$, $s_6 = 0$ and so on, so that

$$s_n = \begin{cases} \frac{1}{(n+1)/2}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

We need to show that $s_n \to 0$ as $n \to \infty$. Indeed, we can bound s_n as follows: $0 \le s_n \le \frac{2}{n+1}$ for all n. Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{2}{n+1} = 0$, by Sandwich Rule $\lim_{n\to\infty} s_n = 0$, as required.

(ii) To test absolute convergence of a series $\sum_{n\ge1} a_n$, we consider the series formed by the absolute values: $\sum_{n\ge1} |a_n|$. Since $|\frac{1}{n}| = |-\frac{1}{n}| = \frac{1}{n}$, we have the series

(*) $$1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \dots$$

The $2n^{\text{th}}$ partial sum of series (*) is $2h_n$ where h_1, $h_2, \dots$ are partial sums of the harmonic series. These partial sums are unbounded — we know that the harmonic series diverges to $+\infty$. Hence series (*) is divergent.

(Alternatively, we can note that $|a_n| \ge \frac{1}{n}$ for all n; since the harmonic series $\sum \frac{1}{n}$ is divergent, series (*) is divergent by the Comparison Test.)

We conclude that the series $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots$ is convergent but not absolutely convergent. Hence it is conditionally convergent.

Extra exercises

Q14. (A question used in a past exam paper.) Determine whether the following series converge. In each case you should briefly justify your answer (in particular, saying what test you are using).

$$(a)\ \sum_{n=1}^{\infty} \frac{5^n}{2^n \cdot n^3} \qquad (b)\ \sum_{n=2}^{\infty} \frac{4n^3}{n^4 - 1} \qquad (c)\ \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}.$$

Q14. Solution. (a) Write the series as $\sum\limits_{n=1}^{\infty} \frac{c^n}{n^3}$ where $c = \frac{5}{2}$. This is a divergent series. Indeed, by a result from MFA, "exponential beats polynomial"; that is, for a given $c > 1$ we have $\frac{n^3}{c^n} \to 0$, equivalently $\frac{c^n}{n^3} \to \infty$ as $n \to \infty$. Therefore, $(\frac{c^n}{n^3})$ is not a null sequence (it does not have limit 0). It follows, that the series $\sum\limits_{n=1}^{\infty} \frac{c^n}{n^3}$ is divergent, by the Nullity Test.

Alternatively, the Ratio Test works in this case. [*The Ratio Test was used to show that* $\frac{n^3}{c^n} \to 0$ *as* $n \to \infty$ *in MFA, yet "exponential beats polynomial" is a very useful fact to remember.*]

(b) $0 \le \frac{1}{n} = \frac{n^3}{n^4} \le \frac{4n^3}{n^4-1}$ for all $n \ge 2$. Since the harmonic series is divergent — hence its tail, $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, too — the series $\sum_{n=2}^{\infty} \frac{4n^3}{n^4-1}$ is divergent by the Comparison Test.

Note: the Ratio Test does not work here because $\ell = 1$.

(c) The series contains both positive and negative terms, so we should try the Alternating Series Test and the Absolute Convergence Theorem.

We note that $\cos(\pi n) = (-1)^n$, so the series is just $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series (multiplied by -1). This series converges by the Alternating Series Test.

Note that this series is not absolutely convergent.

Q15. (A question used in a past exam paper.) Find an $R \in [0, \infty)$ such that the series $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ is absolutely convergent when $|x| < R$ and is divergent when $|x| > R$.

Q15. Solution. By definition of absolute convergence, the given series is absolutely convergent, if, and only if, the series

$$\sum_{n=1}^{\infty} \left| \frac{(n!)^2}{(2n)!} x^n \right|, \text{ same as } \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} |x|^n,$$

is convergent.

If $x = 0$, every term in the series $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} |x|^n$ is 0 and so the series is absolutely convergent. Assume that $x \neq 0$.

It is common to use the Ratio Test in questions of this type. We test the series $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} |x|^n$, which has positive terms, for convergence. We have

$$\ell = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2n+2)!}|x|^{n+1}}{\frac{(n!)^2}{(2n)!}|x|^n} = \lim_{n\to\infty} \frac{((n+1)!)^2(2n)!|x|^{n+1}}{(n!)^2(2n+2)!|x|^n}.$$

Removing the common factor of $(n!)^2(2n)!|x|^n$ from the numerator and the denominator, we simplify this to

$$\ell = \lim_{n\to\infty} \frac{(n+1)^2|x|}{(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2|x|}{(2+\frac{1}{n})(2+\frac{2}{n})} = \frac{(1+0)|x|}{(2+0)(2+0)} = \frac{|x|}{4}.$$

At this point, the Ratio Test tells us that

- if $|x| < 4$, then $\ell < 1$ and so the series is absolutely convergent;
- if $|x| > 4$, then $\ell > 1$ and so the series is divergent.

We conclude that $R = 4$.

(*In the language of power series, the radius of convergence of the given powers series is* 4.)

Q16. Construct non-trivial examples of convergence. Give an example of:

(i) a convergent series $\sum_{n=1}^{\infty} a_n$ such that the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is divergent;

(ii) a convergent series $\sum_{n=1}^{\infty} a_n$ and a sequence $\lambda_n \to 0$ such that the series $\sum_{n=1}^{\infty} \lambda_n a_n$ is divergent;

(iii) a convergent series $\sum_{n=1}^{\infty} a_n$ and nonnegative $\lambda_n \to 0$ such that $\sum_{n=1}^{\infty} \lambda_n a_n$ is divergent.

Q16. Solution. It is useful to observe, for all parts of the question, that any examples of **absolutely convergent** series $\sum_{n=1}^{\infty} a_n$ **will not work.** Indeed, if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and (λ_n) is any bounded sequence, then $\sum_{n=1}^{\infty} \lambda_n a_n$ is absolutely convergent (exercise: prove this). Hence we need to work with conditionally convergent series.

(i) An example, given in the lectures, works: the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent, yet the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. We can put $a_n = \frac{(-1)^{n+1}}{n}$.

(ii) If we make λ_n change the sign, it is easy to break conditional convergence of an alternating series. Take $a_n = \lambda_n = \frac{(-1)^{n+1}}{\sqrt{n}}$. The series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Yet $\lambda_n a_n = \frac{1}{n}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(iii) Recall: the Alternating Series Test works for $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ if the sequence $b_1, b_2, \ldots$ is

- null, i.e., $\lim_{n\to\infty} b_n = 0$;
- decreasing, i.e., $b_n \geq b_{n+1}$ for all n.

If $\lambda_n \geq 0$ and $\lambda_n \to 0$ as $n \to \infty$, the sequence $(\lambda_n b_n)$ will still be null, but may no longer be decreasing. This suggests an example: put

$$a_n = \frac{(-1)^n}{\sqrt{n}}, \qquad \lambda_n = \begin{cases} \frac{1}{\sqrt{n}} & \text{for even } n, \\ 0 & \text{for odd } n. \end{cases}$$

The series $\sum_{n=1}^{\infty} \lambda_n a_n = 0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + \frac{1}{6} + \ldots$ is easily seen to be divergent: its $(2n)^{\text{th}}$ partial sum is $\frac{1}{2} \times$ n^{th} partial sum of the harmonic series. Of course, this example also works in (ii).

Q17. Rearrangements. As shown in Q13, the series $1 - 1 + \dfrac{1}{2} - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{4} + \ldots$ is convergent with sum 0. Now consider its rearrangement

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \frac{1}{7} + \frac{1}{8} - \frac{1}{4} + \ldots$$

where the pattern is, two positive terms followed by one negative term. Show that this rearrangement **does not** have sum 0. (Hint: consider the partial sum s_{3n} and let $n \to \infty$.)

Why does this example not contradict the rearrangement theorems proved in lectures?

Q17. Solution. Write the partial sums of the rearranged series

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \frac{1}{7} + \frac{1}{8} - \frac{1}{4} + \ldots$$

as t_n. To show that the rearranged series does not have sum 0, it is enough to find a subsequence of the sequence (t_n) which does not have limit 0. We take the subsequence t_{3n}:

$$t_3 = 1 + \frac{1}{2} - 1 = 0.5, \quad t_6 = t_3 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = 0.58333\ldots, \quad , \ldots,$$
$$t_{3n} = t_{3(n-1)} + \frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n} = t_{3(n-1)} + \frac{1}{(2n-1)2n} > t_{3(n-1)}.$$

We can see that $(t_{3n})_{n \geq 1}$ is a positive and increasing sequence. All terms of this subsequence are greater than or equal to 0.5. It follows that the rearranged series, if converges, must converge to a sum greater than 0.5, and cannot converge with sum 0.

The rearrangement theorems, proved in the course, require the series to be non-negative or absolutely convergent. Yet the given series is conditionally convergent. The sum can change if the series is rearranged; moreover, Riemann's rearrangement theorem tells us that for any real number s, the series can be rearranged so at to have sum s.

Comment: In fact, it is not difficult to show that the rearranged series $1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \frac{1}{7} + \frac{1}{8} - \frac{1}{4} + \ldots$ is convergent. A bit more work is needed to show that the sum is $\ln(2) = 0.693\ldots$