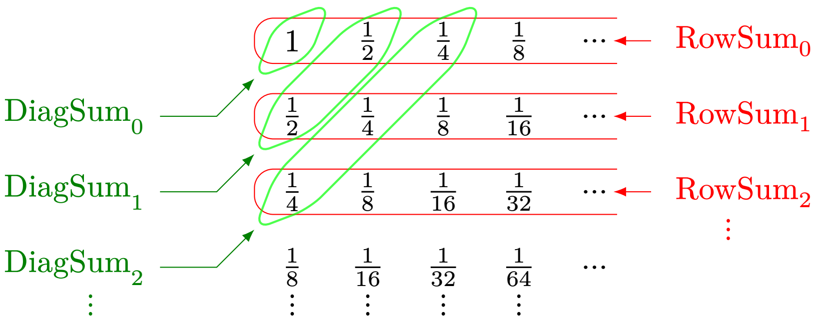
MATH11112 Real Analysis (2025). Exercise sheet for week 03 Convergence tests

Homework

Attempt all parts of Q9 and submit your work online before 2pm on Tuesday 11th February. You will find the Gradescope submission link in the Week 3 folder on Blackboard.

Q9. Consider the double series with general term given by $a_{m,n} = \dfrac{1}{2^{m+n}}$ for $m, n \geq 0$. Let RowSum_m denote the sum $\sum_{n=0}^{\infty} a_{m,n}$ of the entries in the mth row, and $\mathrm{DiagSum}_d$ denote the sum $\sum\limits_{m+n=d} a_{m,n}$ of the entries on the dth diagonal:



(i) Write down a formula expressing RowSum_m in terms of m. (*Hint:* each row of the given double series is a geometric series.)

(ii) Let $S = \sum_{m=0}^{\infty} \mathrm{RowSum}_m$; find the numerical value of S.

(iii) Write down a formula expressing $\mathrm{DiagSum}_d$ in terms of d.

(iv) Since the given double series has non-negative terms, a result from the lectures says that $\sum_{d=0}^{\infty} \mathrm{DiagSum}_d = S$. Use this result to calculate $\sum\limits_{n=0}^{\infty} \dfrac{n}{2^n}$, briefly explaining what you do.

Supervision work

Attempt these questions and bring your solutions to the Week 3 supervision class for discussion.

Q10. Exercise on the Ratio Test. The **Fibonacci sequence** $F_1, F_2, F_3, \ldots$ is defined by the rule $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_{n-1} + F_n$ for all $n \geq 2$. Consider the **Fibonacci reciprocal series**

$$\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{F_n}.$$

(i) Write down the first six terms of the series.

(ii) Write down the explicit formula for F_n in terms of the two roots, α and β, of the quadratic equation $x^2 = x + 1$. This formula should be known to you from Section 2 of MFA (Semester 1).

(iii) Apply the Ratio Test to the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{F_n}$. Use the formula from (ii) to find the limit ℓ needed for the test. Then, based on the numerical value of ℓ, determine whether the series is convergent.

Q11. Exercises on the Comparison Test. (i) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent, by comparing it with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(ii) Show that $\sum_{n=1}^{\infty} \frac{100}{n^2 - 0.5\sqrt{n}}$ is convergent, by comparing it with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, multiplied by a suitable constant.

(iii) Are $\sum_{n=1}^{\infty} \frac{1}{n-0.5}$ and $\sum_{n=1}^{\infty} \frac{1}{n+100.5}$ convergent or divergent?

Q12. (a question from a past exam paper) Use Algebra of Infinite Sums to prove that the following series is convergent: $\sum_{n=1}^{\infty} \Big(\frac{1000}{n^2} + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \Big)$.

Q13. Absolute vs. conditional convergence. (i) Show: $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \ldots$ is a convergent series with sum 0.

(ii) Determine whether the series in (i) is **absolutely convergent** or **conditionally convergent.** (By definition, "conditionally convergent" means convergent but not absolutely convergent.)

Extra exercises

Attempt these questions in your own time and compare your answers with the model solutions published on Monday in week 4. Some of these questions may be discussed in the Examples Class on Thursday in week 4.

Q14. (A question used in a past exam paper.) Determine whether the following series converge. In each case you should briefly justify your answer (in particular, saying what test you are using).

$$(a) \sum_{n=1}^{\infty} \frac{5^n}{2^n \cdot n^3} \qquad (b) \sum_{n=2}^{\infty} \frac{4n^3}{n^4 - 1} \qquad (c) \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}.$$

Q15. (A question used in a past exam paper.) Find an $R \in [0, \infty)$ such that the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ is absolutely convergent when $|x| < R$ and is divergent when $|x| > R$.

Q16. Construct non-trivial examples of convergence. Give an example of:

(i) a convergent series $\sum_{n=1}^{\infty} a_n$ such that the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is divergent;

(ii) a convergent series $\sum_{n=1}^{\infty} a_n$ and a sequence $\lambda_n \to 0$ such that the series $\sum_{n=1}^{\infty} \lambda_n a_n$ is divergent;

(iii) a convergent series $\sum_{n=1}^{\infty} a_n$ and nonnegative $\lambda_n \to 0$ such that $\sum_{n=1}^{\infty} \lambda_n a_n$ is divergent.

Q17. Rearrangements. As shown in Q13, the series $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \ldots$ is convergent with sum 0. Now consider its rearrangement

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \frac{1}{7} + \frac{1}{8} - \frac{1}{4} + \ldots$$

where the pattern is, two positive terms followed by one negative term. Show that this rearrangement **does not** have sum 0. (Hint: consider the partial sum s_{3n} and let $n \to \infty$.)

Why does this example not contradict the rearrangement theorems proved in lectures?