MATH11112 Real Analysis (2025). Exercise sheet for week 01 Series: first examples. Series with non-negative terms — SOLUTIONS

Supervision work

Q1. Simple examples of infinite series. Which of the following series are convergent and which are divergent? Refer to the definition to justify your answer.

(i) $0+0+0+\dots$

(ii) $1+1+1+\dots$

(iii) $\sum_{n=0}^{\infty}(-1)^n$, same as $1+(-1)+1+(-1)+\dots$

(iv) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

[**Hint**: there is a nice formula for the nth partial sum s_n in (i)–(iii) but not in (iv). In (iv), the fact that the sum of n terms is at least $n \times$ (smallest term) allows you to bound s_n from below.]

Does the fact that $1+(-1)=0$ imply that series (iii) is convergent with sum 0?

Q1. Solution. In (i)–(iii), we can calculate the nth partial sum $s_n = a_1 + \dots + a_n$.

(i) $s_n = 0$ for all n, so $(s_n)_{n\ge 1}$ is a constant sequence which has limit 0. Hence the series $0+0+0+\dots$ is convergent with sum 0.

(ii) $s_n = n$ so $s_n \to +\infty$. We say that the series $1+1+1+\dots$ **diverges to infinity**, which means that the sequence of partial sums diverges to infinity. We write $\sum_{n=1}^{\infty} 1 = +\infty$.

(iii) [A variant of series notation: we number the terms starting from $n=0$.]

We see that $s_0 = 1$, $s_1 = 0$, $s_2 = 1$, in other words $s_n = 1$ if n is even and $s_n = 0$ if n is odd. The same can be written as $s_n = \frac{1+(-1)^n}{2}$.

We recognise the sequence $(s_n)_{n\ge 0} = 1, 0, 1, 0, \dots$ as an example of a sequence which does not have a limit (from MFA). Hence, by definition, the series $\sum_{n=0}^{\infty}(-1)^n$ is not convergent. A series which is not convergent is **divergent.**

This shows that **grouping the terms** (e.g., replacing $1+(-1)$ by 0) in a series that does not converge could lead to a convergent series.

We will later see examples of convergent series in which **putting the terms in a different order** leads to a convergent series with a **different sum**, or to a divergent series. This shows that we need to be strict about the definition of the sum of a series $\sum_{n=1}^{\infty} a_n$ as the limit of the sequence $a_1,\ a_1+a_2,\ a_1+a_2+a_3, \dots$: in general, rearrangements which we can use in **finite** sums are not allowed in infinite series.

(iv) There is no “nice” formula for the nth partial sum s_n here — as is most often the case when studying series! The theory developed in the course aims to test series for convergence without being able to write down a formula for partial sums. Real Analysis is very much about **inequalities**. We can easily estimate s_n from below:

$$s_n = 1 + \tfrac{1}{\sqrt{2}} + \dots + \tfrac{1}{\sqrt{n}} \quad \geq \quad n \times \tfrac{1}{\sqrt{n}} = \sqrt{n}.$$

Since $s_n \geq \sqrt{n}$ for all n and $\sqrt{n} \to +\infty$, we conclude that $s_n \to +\infty$. We can write this as "$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = +\infty$".

Q2. A "telescoping" series. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

(i) Calculate the partial sums s_1, s_2, s_3. Then show that the nth partial sum is $s_n = 1 - \frac{1}{n+1}$.

(ii) Is the series convergent? If so, what is the sum of this series?

Q2. Solution.

(i) We have $s_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$, $s_2 = s_1 + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, $s_3 = s_2 + \frac{1}{3\cdot 4} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$. Noticing a pattern, we can conjecture that $s_n = \frac{n}{n+1}$ for all n, then prove our conjecture by induction. [You can write out a formal proof by induction, for practice.]

A standard trick, though, which hides the induction behind the dots "... " is

$$\tfrac{1}{n(n+1)} = \tfrac{1}{n} - \tfrac{1}{n+1} \quad \Longrightarrow \quad s_n = \tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \dots + \tfrac{1}{n(n+1)} = (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{2}{3}) + \dots + (\tfrac{1}{n} - \tfrac{1}{n+1}).$$

We may rearrange terms in this **finite sum** and obtain, after cancellations, that $s_n = 1 - \frac{1}{n+1}$.

(ii) Since $1 - \frac{1}{n+1} \to 1$ as $n \to \infty$, we conclude that the series is convergent with sum 1.

Comment: sums in which all intermediate terms cancel like the s_n above, are sometimes called "telescoping sums" — imagine collapsing a vintage telescope or "spyglass". Hence this series may be called a telescoping series.

Q3. The series of inverse squares and the Basel problem. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(i) By comparing with $\sum \frac{1}{n(n+1)}$ (Q2), show that $s_n = 1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}$ is less than 2 for all n.

(ii) Give a reason why $s_1, s_2, s_3, \dots$ is an **increasing** sequence.

(iii) State the key result about convergence of a **increasing sequence that is bounded above.**

(iv) Use (i), (ii) and (iii) to conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent with sum $s \leq 2$.

Comment: There is no nice formula for s_n here, yet in 1734 Euler found that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \; = \frac{\pi^2}{6},$$

solving what was known as the Basel problem. Having taken MATH11112 Real Analysis, you will be able to follow some proofs of this fact; other proofs use Complex Analysis. Can you see that $\frac{\pi^2}{6} \leq 2$?

Q3. Solution.

(i) There are two easy ways to compare:

Way 1. Notice that $\frac{1}{n^2} < \frac{1}{(n-1)n}$ for all $n \geq 2$. It follows that

$$s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 + \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n}\right).$$

By Q2, the right-hand side is $1 + \left(1 - \frac{1}{n}\right)$. Hence $s_n < 2$ for all $n \geq 2$.

Way 2. Notice that $\frac{n+1}{2} \leq n$ for all $n \geq 1$, so $\frac{2}{n+1} \geq \frac{1}{n}$ and $\frac{1}{n^2} \leq \frac{2}{n(n+1)}$. We have

$$s_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2\left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}\right) = 2(1 - \frac{1}{n+1}) < 2.$$

(ii) Recall from MFA that a sequence $(s_n)_{n \geq 1}$ is said to be **increasing**, if $s_n \leq s_{n+1}$ for all n.

If $s_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ and $s_{n+1} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(n+1)^2}$, we have $s_{n+1} - s_n = \frac{1}{(n+1)^2}$ which is non-negative. Hence $s_{n+1} - s_n \geq 0$ and so $s_n \leq s_{n+1}$.

This way, it is easy to see that for every **series with non-negative terms**, the partial sums form an increasing sequence.

(iii) We quote the following result from MFA:

> Proposition. Let (a_n) be a sequence of real numbers. Suppose that (a_n) is an increasing sequence that is bounded above. Then (a_n) is convergent.

(iv) $s_n \leq 2$ for all n as shown in (i), so by definition of "bounded above", $(s_n)_{n \geq 1}$ is a bounded above sequence.

By (ii), $(s_n)_{n \geq 1}$ is an increasing sequence.

Hence we apply the Proposition from MFA and conclude that (s_n) is a convergent sequence; i.e., there is a real number s such that $s_n \to s$ as $n \to \infty$. By definition of a convergent series, this means that the series $\sum_{n \geq 1} \frac{1}{n^2}$ is convergent with sum s.

We quote another result from MFA:

> Corollary. Suppose that (a_n) and (b_n) are sequences of real numbers. Suppose that $a_n \leq b_n$ for all n. Suppose that $a_n \to a$, $b_n \to b$ as $n \to \infty$. Then $a \leq b$.

Since $s_n \leq 2$ for all n and $s_n \to s$, we use the Corollary to conclude that $s \leq 2$.

Comparison with Euler's result: $\frac{\pi^2}{6} = 1.6449\ldots$ is less than 2.

Q4. Geometric series and infinite decimals example. You need to know the formulae for the sum $a + ar + \dots + ar^n$ and the geometric series sum $\sum_{n=0}^{\infty} ar^n$ if $|r| < 1$; review them now.

(i) Let $a_1, a_2, \dots$ be a sequence of decimal digits, i.e., $a_n \in \{0, 1, \dots, 9\}$ for all n. The real number represented by the infinite decimal $0.a_1 a_2 a_3 \dots$ is defined to be the sum of the series

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \dots = \sum_{n=1}^{\infty} \frac{a_n}{10^n}.$$

Use the sum of geometric series to write the real numbers expressed by the infinite decimals $0.5555\dots$ and $0.9999\dots$ as rationals p/q where $p, q \in \mathbb{N}$.

(ii) Likewise, calculate $\displaystyle\sum_{n=1}^{\infty} \frac{428571}{10^{6n}}$.

(iii) Show that any series of the form given in (i) is **convergent,** with sum between 0 and 1. [*Hint*: the series has non-negative terms; to show convergence, you may bound the partial sums from above.]

Q4. Solution. As seen in MFA and again in Real Analysis, if $r \neq 1$, $a + ar + \dots + ar^n = a\frac{1-r^{n+1}}{1-r}$. If $|r| < 1$, then $r^{n+1} \to 0$ as $n \to \infty$ and so the sum of geometric series with initial term a and ratio r is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

(i) We have $0.5555\dots = \frac{5}{10} + \frac{5}{10^2} + \dots$ which is the sum of geometric series with $a = \frac{5}{10}$ and $r = \frac{1}{10}$. This sum is $\frac{5/10}{1-1/10} = \frac{5/10}{9/10} = 5/9$.

In $0.9999\dots$ we have $a = \frac{9}{10}$, $r = \frac{1}{10}$ and so the sum is $\frac{9/10}{1-9/10} = \frac{9/10}{9/10} = 1$.

(ii) $a = 4218571/10^6$, $r = 1/10^6$ so the sum is $\frac{4218571/10^6}{1-1/10^6} = \frac{428571}{999999}$. Since $428571 = 142857 \times 3$ and $999999 = 142857 \times 7$, the answer simplifies to $3/7$.

(iii) Since $0 \leq \frac{a_k}{10^k} \leq \frac{9}{10^k}$ for all k, the nth partial sum $s_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \dots + \frac{a_n}{10^n} \leq \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n}$. This is a partial sum of $\sum_{n=1}^{\infty} \frac{9}{10^n} = 1$. Hence $s_1, s_2, \dots$ is an increasing sequence of non-negative numbers bounded above by 1; it must then have a non-negative limit bounded above by 1.

Extra exercises

Attempt these questions in your own time and compare your answers with the model solutions published on Monday in week 2. Some of these questions will be discussed in the Examples Class on Thursday in week 2.

Q5. The number e is irrational. (i) By showing that partial sums form an increasing sequence and are bounded above by 3, prove that the following series is convergent:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Notation: the sum of the series $\sum\limits_{k=0}^{\infty} \dfrac{1}{k!}$ is denoted e.

(ii) Let s_n be the partial sum $1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dots + \dfrac{1}{n!}$. Show that $s_n < e < s_n + \dfrac{1}{n(n!)}$.

(iii) Prove that e is an irrational number.

Hint: deduce from (ii) that $n!\,s_n < n!\,e < n!\,s_n + 1$ for all n. Note that $n!\,s_n$ is an integer for all $n \in \mathbb{N}$ (*why?*) and so $n!\,s_n$, $n!\,s_n + 1$ are two consecutive integers. Assuming that e is rational, let $n \in \mathbb{N}$ be such that $n!\,e$ is an integer (*why does such an n exist?*) and arrive at a contradiction.

Q5. Solution. (i) $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \geq 1 \cdot 2 \cdot 2 \cdot \ldots \cdot 2$, that is, $n! \geq 2^{n-1}$ for all $n \in \mathbb{N}$. Hence $s_n \leq 1 + \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} = 3 - \frac{1}{2^{n-1}}$. The sequence $s_1, s_2, \ldots$ of partial sums is increasing (because s_{n+1} is obtained by adding a non-negative number to s_n) and bounded above by 3, hence the given series with non-negative terms converges with sum $e \leq 3$.

(ii) We have $s_n < s_{n+1}$ and, for all k, $s_k \leq e = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n$. So $s_n < s_{n+1} \leq e$. On the other hand,

$$e = s_n + \tfrac{1}{(n+1)!} + \tfrac{1}{(n+2)!} + \tfrac{1}{(n+3)!} + \dots < s_n + \tfrac{1}{(n+1)!} + \tfrac{1}{(n+1)!(n+1)} + \tfrac{1}{(n+1)!(n+1)^2} + \ldots$$

So $e - s_n$ is bounded above by the sum of geometric series with initial term $\frac{1}{(n+1)!}$ and ratio $\frac{1}{n+1}$. This sum is $\frac{1}{(n+1)!}\frac{1}{1-\frac{1}{n+1}} = \frac{1}{(n+1)!-n!} = \frac{1}{n(n!)}$.

(iii) Assume for contradiction that e is rational. Then $e = \frac{m}{n}$ for some $m, n \in \mathbb{N}$, so ne is an integer, and then $n!\,e = (n-1)!(ne)$ is also an integer.

Multiplying all parts of the inequality $s_n < e < s_n + \frac{1}{n(n!)}$ by $n!$, we have $n!\,s_n < n!\,e < n!\,s_n + \frac{1}{n}$. Note that $n!\,s_n = \frac{n!}{1} + \frac{n!}{2!} + \dots + \frac{n!}{n!}$ is an integer: indeed, $\frac{n!}{k!} = (k+1)(k+2)\ldots n$ is an integer for all $k \leq n$, and a sum of several integers is an integer.

Hence the integer $n!\,e$ is strictly between two consecutive integers $n!\,s_n$, $n!\,s_n + 1$. This is impossible. The contradiction arose because we assumed that e was rational, so e must be irrational.

Q6. (A question used in a past exam paper.) If $e^{1/2}$ is sum of the series $\sum\limits_{r=0}^{\infty} \dfrac{1}{2^r\, r!}$, prove that

$$e^{1/2} - \Big(1 + \frac{1}{2} + \frac{1}{8}\Big) \leq \frac{1}{24}.$$

Q6. Solution. We have $e^{1/2} = \sum\limits_{r=0}^{\infty} \dfrac{1}{2^r\, r!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{2^3\,3!} + \ldots$ so $e^{1/2} - \big(1 + \frac{1}{2} + \frac{1}{8}\big)$ is the sum of the series $\sum\limits_{r=3}^{\infty} \dfrac{1}{2^r\, r!}$.

When $r \geq 3$, one has $\dfrac{1}{2^r\, r!} \leq \dfrac{1}{6 \times 2^r}$ as $r! \geq 6$.

Hence, using the sum of geometric series,

$$\sum_{r=3}^{\infty} \frac{1}{2^r\, r!} \leq \frac{1}{6}\Big(\frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots\Big) = \frac{1}{6} \times \frac{2}{2^3} = \frac{1}{24},$$

as claimed.

Q7. The Cauchy condensation test. (i) Try to prove the following theorem (or work through a proof given in the literature):

Suppose $a_1, a_2, a_3 \dots$ are non-negative reals such that $a_1 \geq a_2 \geq a_3 \geq \dots$. The series

(a) $$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

is convergent if, and only if, the "condensed" series

(b) $$a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots = \sum_{k=0}^{\infty} 2^k a_{2^k}$$

is convergent.

Hint: let s_n be partial sums of series (a) and t_k be partial sums of series (b). Show that $s_{2^n} \leq t_n \leq 2s_{2^n}$. So if the sequence (s_n) is bounded, (t_k) must also be bounded, and vice versa.

(ii) Apply the above theorem (known as the Cauchy condensation test) to show that the harmonic series $\sum_{n\geq 1} \frac{1}{n}$ is divergent. (Don't forget: the test is applicable only to non-negative series where $a_1, a_2, a_3 \dots$ is decreasing.)

Q7. Solution. (i) Consider the following table with three rows and 2^n columns:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	...	a_{2^n-1}	a_{2^n}	row sum $= s_{2^n}$
a_1	a_2	a_2	a_4	a_4	a_4	a_4	a_8	a_8	...	$a_{2^{n-1}}$	a_{2^n}	row sum $= t_{n-1} + a_{2^n}$
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4	a_5	...	$a_{2^{n-1}}$	$a_{2^{n-1}}$	row sum $= 2s_{2^{n-1}}$

The top row contains the terms of the original series (a). The middle row is formed by repeating each a_{2^k} a total of 2^k times. The bottom row is obtained by repeating each a_k twice.

It is easy to see that in each column, the entries (non-strictly) increase. We therefore have inequalities between row sums: $s_{2^n} \leq t_{n-1}+a_{2^n}$, which implies $s_{2^n} \leq t_n$, and $t_{n-1}+a_{2^n} \leq 2s_{2^{n-1}}$, which implies $t_{n-1} \leq 2s_{2^{n-1}}$.

Now, if series (b) is convergent with sum T, then $t_n \leq T$ for all n, and it follows that $s_n \leq T$ for all n (as $s_n \leq s_{2^n} \leq t_n$), so the increasing sequence (s_n) is bounded above hence has a limit. Thus, convergence of (b) implies convergence of (a).

The other way around, if (a) is convergent with sum S, then the partial sums of (b) are bounded above by $2S$, and so (b) is convergent.

(ii) The Cauchy condensation test is applicable to the harmonic series because $1 \geq \frac{1}{2} \geq \frac{1}{3} \geq \dots$. The condensed series of the harmonic series $(a_n = \frac{1}{n})$ is

$$a_1 + 2a_2 + 4a_4 + \cdots = 1 + 2 \times \frac{1}{2} + 4 \times \frac{1}{4} + \cdots = 1 + 1 + 1 + \dots$$

which diverges to $+\infty$. Hence, by the Cauchy condensation test, the harmonic series is divergent.

Q8. Riemann's zeta function. Let s be a positive real number. Use the Cauchy condensation test from Q7 to show that

(i) if $s > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^s} < +\infty$; (ii) if $s \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^s} = +\infty$.

Comment: the zeta function $\zeta(s)$ is defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s > 1$. For example, $\zeta(2) = \frac{\pi^2}{6}$, by a result of Euler. It turns out that $\zeta(s)$ can be extended to real numbers $s < 1$ (by a different formula) and moreover to all complex numbers $s \neq 1$ as a continuous, and differentiable, function of s. The **Riemann Hypothesis** is a statement about this function $\zeta(s)$ and is one of the most famous unsolved problems in mathematics.

Q8. Solution. The Cauchy condensation test is applicable because $\frac{1}{1^s} \geq \frac{1}{2^s} \geq \frac{1}{3^s} \geq \dots$. If $a_n = \frac{1}{n^s}$, the condensed series is

$$\sum_{k\geq 0} 2^k a_{2^k} = \sum_{k\geq 0} 2^k \frac{1}{(2^k)^s} = \sum_{k\geq 0} 2^{k-ks} = \sum_{k\geq 0} (2^{1-s})^k.$$

This is a **geometric** series with ratio $r = 2^{1-s}$. If $s > 1$, then $r < 1$ so that the geometric series is convergent, hence by the Cauchy condensation test $\sum \frac{1}{n^s}$ is convergent. If $s \leq 1$, we have $r \geq 1$, meaning that the condensed geometric series is divergent, and so the series $\sum \frac{1}{n^s}$ must, too, be divergent.