MATH11112 Real Analysis (2025). Exercise sheet for week 01

Series: first examples. Series with non-negative terms

There is no homework to hand in this week.

Supervision work

Attempt these questions and bring your solutions to your Week 1 supervision class for discussion.

Definition (from week 1 lectures). For real numbers a_n, an **infinite series** is an expression of the form $\sum_{n=1}^{\infty} a_n$ (also written $a_1 + a_2 + \dots + a_n + \dots$, $\sum_{n\geq 1} a_n$ or just $\sum a_n$).

- The n^{th} **partial sum** of this series is $s_n = a_1 + \dots + a_n = \sum_{i=1}^{n} a_i$.
  (Terms of the series might be numbered from 0 or from another integer, but s_n is the sum **up to and including the** n**th term**, e.g., $s_2 = a_0 + a_1 + a_2$)
- If the sequence of partial sums has a limit: $s_n \to s$ as $n \to \infty$, we say that the series $\sum_{n=1}^{\infty} a_n$ is **convergent with sum** s, and write $\sum_{n=1}^{\infty} a_n = s$.
- A series which is not convergent is said to be **divergent.**

Q1. Simple examples of infinite series. Which of the following series are convergent and which are divergent? Refer to the definition to justify your answer.

(i) $0 + 0 + 0 + \dots$

(ii) $1 + 1 + 1 + \dots$

(iii) $\sum_{n=0}^{\infty} (-1)^n$, same as $1 + (-1) + 1 + (-1) + \dots$

(iv) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

[**Hint**: there is a nice formula for the nth partial sum s_n in (i)–(iii) but not in (iv). In (iv), the fact that the sum of n terms is at least $n \times$ (smallest term) allows you to bound s_n from below.]

Does the fact that $1 + (-1) = 0$ imply that series (iii) is convergent with sum 0?

Q2. A "telescoping" series. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

(i) Calculate the partial sums s_1, s_2, s_3. Then show that the nth partial sum is $s_n = 1 - \frac{1}{n+1}$.

(ii) Is the series convergent? If so, what is the sum of this series?

Q3. The series of inverse squares and the Basel problem. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(i) By comparing with $\sum \frac{1}{n(n+1)}$ (Q2), show that $s_n = 1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}$ is less than 2 for all n.

(ii) Give a reason why $s_1, s_2, s_3, \dots$ is an **increasing** sequence.

(iii) State the key result about convergence of a **increasing sequence that is bounded above.**

(iv) Use (i), (ii) and (iii) to conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent with sum $s \leq 2$.

Comment: There is no nice formula for s_n here, yet in 1734 Euler found that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6},$$

solving what was known as the Basel problem. Having taken MATH11112 Real Analysis, you will be able to follow some proofs of this fact; other proofs use Complex Analysis. Can you see that $\frac{\pi^2}{6} \le 2$?

Q4. Geometric series and infinite decimals example. You need to know the formulae for the sum $a + ar + \dots + ar^n$ and the geometric series sum $\sum_{n=0}^{\infty} ar^n$ if $|r| < 1$; review them now.

(i) Let $a_1, a_2, \dots$ be a sequence of decimal digits, i.e., $a_n \in \{0, 1, \dots, 9\}$ for all n. The real number represented by the infinite decimal $0.a_1a_2a_3\dots$ is defined to be the sum of the series

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \dots = \sum_{n=1}^{\infty} \frac{a_n}{10^n}.$$

Use the sum of geometric series to write the real numbers expressed by the infinite decimals $0.5555\dots$ and $0.9999\dots$ as rationals p/q where $p, q \in \mathbb{N}$.

(ii) Likewise, calculate $\displaystyle\sum_{n=1}^{\infty} \frac{428571}{10^{6n}}$.

(iii) Show that any series of the form given in (i) is **convergent,** with sum between 0 and 1. [*Hint*: the series has non-negative terms; to show convergence, you may bound the partial sums from above.]

Extra exercises

Attempt these questions in your own time and compare your answers with the model solutions published on Monday in week 2. Some of these questions will be discussed in the Examples Class on Thursday in week 2.

Q5. The number e is irrational. (i) By showing that partial sums form an increasing sequence and are bounded above by 3, prove that the following series is convergent:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Notation: the sum of the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ is denoted e.

(ii) Let s_n be the partial sum $1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$. Show that $s_n < e < s_n + \frac{1}{n(n!)}$.

(iii) Prove that e is an irrational number.

Hint: deduce from (ii) that $n!\, s_n < n!\, e < n!\, s_n + 1$ for all n. Note that $n!\, s_n$ is an integer for all $n \in \mathbb{N}$ (*why?*) and so $n!\, s_n$, $n!\, s_n + 1$ are two consecutive integers. Assuming that e is rational, let $n \in \mathbb{N}$ be such that $n!\, e$ is an integer (*why does such an n exist?*) and arrive at a contradiction.

Q6. (A question used in a past exam paper.) If $e^{1/2}$ is sum of the series $\sum\limits_{r=0}^{\infty} \dfrac{1}{2^r\, r!}$, prove that

$$e^{1/2} - \left(1 + \frac{1}{2} + \frac{1}{8}\right) \leq \frac{1}{24}.$$

Q7. The Cauchy condensation test. (i) Try to prove the following theorem (or work through a proof given in the literature):

Suppose $a_1, a_2, a_3 \ldots$ are non-negative reals such that $a_1 \geq a_2 \geq a_3 \geq \ldots$. The series

(a) $$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

is convergent if, and only if, the "condensed" series

(b) $$a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots = \sum_{k=0}^{\infty} 2^k a_{2^k}$$

is convergent.

Hint: let s_n be partial sums of series (a) and t_k be partial sums of series (b). Show that $s_{2^n} \leq t_n \leq 2s_{2^n}$. So if the sequence (s_n) is bounded, (t_k) must also be bounded, and vice versa.

(ii) Apply the above theorem (known as the Cauchy condensation test) to show that the harmonic series $\sum_{n\geq 1} \frac{1}{n}$ is divergent. (Don't forget: the test is applicable only to non-negative series where $a_1, a_2, a_3 \ldots$ is decreasing.)

Q8. Riemann's zeta function. Let s be a positive real number. Use the Cauchy condensation test from Q7 to show that

(i) if $s > 1$, then $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^s} < +\infty$; (ii) if $s \leq 1$, then $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^s} = +\infty$.

Comment: the zeta function $\zeta(s)$ is defined by $\zeta(s) = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^s}$ for $s > 1$. For example, $\zeta(2) = \frac{\pi^2}{6}$, by a result of Euler. It turns out that $\zeta(s)$ can be extended to real numbers $s < 1$ (by a different formula) and moreover to all complex numbers $s \neq 1$ as a continuous, and differentiable, function of s. The **Riemann Hypothesis** is a statement about this function $\zeta(s)$ and is one of the most famous unsolved problems in mathematics.