MATH11112 REAL ANALYSIS Week 05 lecture B

SEAtS 518405

[Organisation of the course]

[Today is the final lecture of Y.B.'s part of the course]

[Mark Coleman takes over on Monday in Week 6]

[Y.B. will set a Coursework Test (online)

timed at 30 min, one attempt only,

to be submitted between

11am Thurs 06 Mar – 11am Fri 07 Mar,

+ Mock Test (online) to practise,

– look for announcements]

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MATH11112 REAL ANALYSIS **Week** 05 **lecture** B

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* We have proved theorems known as "rules of differentiation"

* We have shown that
  - a polynomial is everywhere diff^able (by Sum and Product Rule);
  - e^x is everywhere diff^able;
  - a rational function $\left(= \frac{P(x)}{Q(x)}, P, Q \text{ polynomials}\right)$ is diff^ble everywhere it is defined (where $Q(x) \neq 0$) (by Quotient Rule);
  - $\ln(x)$ is diff^ble where defined (Inverse Rule)
  - sums, products, compositions of the above are diff^ble.

* WHAT ABOUT sin AND cos ?

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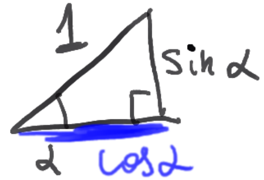
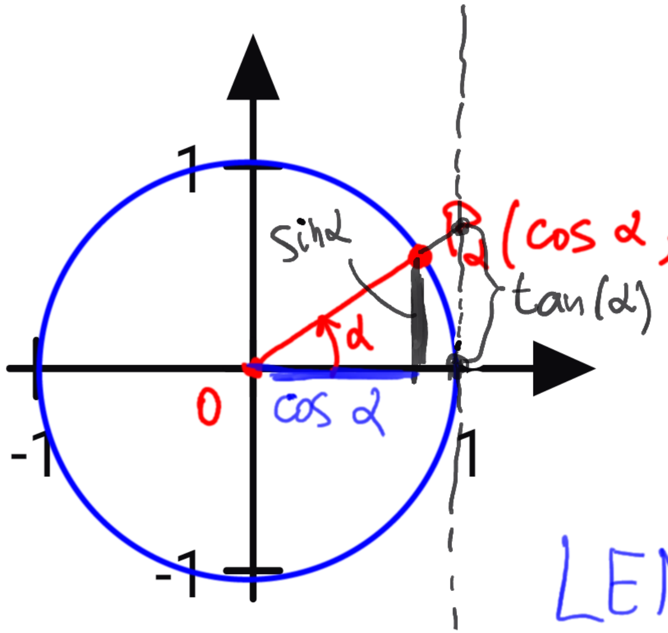
Definition of sin and cos

$\alpha \in \left(0, \frac{\pi}{2}\right)$





$P_\alpha(\cos\alpha, \sin\alpha) \leftarrow$ this is the def^n of $\sin\alpha$ and $\cos\alpha$

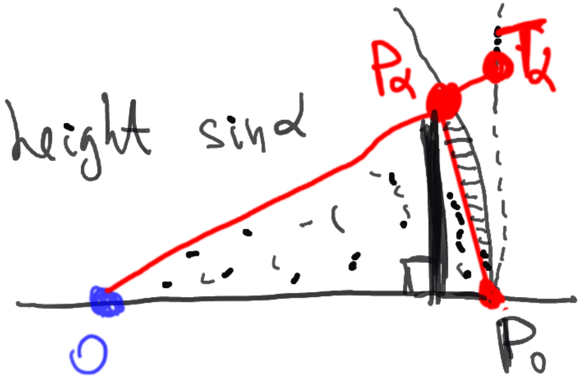
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \quad (\text{when } \cos\alpha \neq 0)$$

LEM $\alpha \in \left(0, \frac{\pi}{2}\right) \Rightarrow 0 \leq \sin\alpha \leq \alpha \leq \tan\alpha.$

Pf

ΔOP_0P_α has height $\sin\alpha$

base $= 1$



$$\text{AREA}(\Delta OP_0P_\alpha) = \frac{1}{2}\,\text{base} \times \text{height} = \frac{1}{2}\sin\alpha$$

$$\text{AREA}(\text{sector } OP_0P_\alpha) = \frac{1}{2}\alpha$$

$$\text{AREA}(\Delta OP_0T_\alpha) = \frac{1}{2}\tan\alpha$$

So, $\frac{1}{2}\sin\alpha \leq \frac{1}{2}\alpha \leq \frac{1}{2}\tan\alpha$. ∎

COR $\lim_{x\to 0} |\sin x| = 0.$

Pf When $x \in (0, \frac{\pi}{2})$, we have $0 \le |\sin x| \le x$
$x \to 0+$:

So by Sandwich Rule,

$$\lim_{x\to 0+} |\sin x| = 0.$$

$$\lim_{x\to 0-} |\sin x| \underset{t=-x}{=\!=} \lim_{t\to 0+} |\sin(-t)| = \lim_{t\to 0+} |\sin t| = 0.$$

$|\sin(-t)| = |-\sin t| = |\sin t|$

Both $\lim_{x\to 0+} |\sin x|$ and $\lim_{x\to 0-} |\sin x|$ exist and are equal $\Rightarrow \lim_{x\to 0} |\sin x| = 0.$ $\blacksquare$

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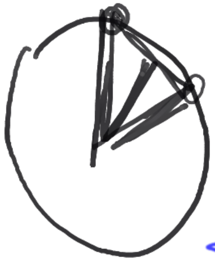
LEM (sine and cosine subtraction formulas)

(i) $\sin y - \sin x = 2 \sin \frac{y-x}{2} \cos \frac{y+x}{2}$

(ii) $\cos y - \cos x = -2 \sin \frac{y-x}{2} \sin \frac{\overset{y+x}{\cancel{y-x}}}{2}$

Pf  (geometry) − omitted. ∎

PROP $\sin x$ and $\cos x$ are continuous fn's on $\mathbb{R}$.

Pf $|\sin y - \sin x| = 2\left|\sin \frac{y-x}{2}\right| \overbrace{\left|\cos \frac{y+x}{2}\right|}^{\leq 1} \leq 2|\sin h|$

where $h = \frac{y-x}{2}$.

So, $-2|\sin h| \leq \sin y - \sin x \leq 2|\sin h|$

(opening out)

$\lim_{y \to x}$: $\boxed{h \to 0}$

$$-2|\sin h| \le \sin y - \sin x \le 2|\sin h|$$

(Lemma)

$$0$$

By Sandwich Rule, $\lim_{y \to x} \sin y - \sin x = 0$

Equivalently (AoL) $\lim_{y \to x} \sin y = \sin x$

So sin is continuous at x (arbitrary) by the Criterion of Continuity.

<u>Similarly</u> cos is continuous at x (using $\cos y - \cos x$) ∎

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PROP (Special Limit for sine) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1.$

[Informal example: $x = \pi/6 \approx 0.502$
$\sin x = 0.500$]

Pf $x \in (0, \pi/2)$

$$0 < \sin x \leq x \leq \tan x \quad \text{(take reciprocals)}$$

$$\Rightarrow \quad \frac{1}{\sin x} \geq \frac{1}{x} \geq \frac{\cos x}{\sin x} \quad (\times \sin x)$$

$$\Rightarrow \quad 1 \geq \frac{\sin x}{x} \geq \cos x$$

$\cos x$ is cont's so $\lim\limits_{x\to 0+} \cos x = \cos 0 = 1$

$\boxed{\lim\limits_{x\to 0+} :}$ 1

By Sandwich, $\lim\limits_{x\to 0+} \frac{\sin x}{x} = 1.$

$$\lim_{x \to 0-} \frac{\sin x}{x} \underset{y=-x}{=} \lim_{y \to 0+} \frac{\sin(-y)}{(-y)} = \lim_{y \to 0+} \frac{-\sin y}{-y} = \underline{1}$$

by the previous limit.

Since $\lim\limits_{x \to 0+}$ and $\lim\limits_{x \to 0-}$ exist and equal 1,

$\lim\limits_{x \to 0} \frac{\sin x}{x}$ exists and equals 1. ∎

THM $\sin x$, $\cos x$ are diff^{ble} everywhere on $\mathbb{R}$,

$\sin' x = \cos x$, $\quad \cos' x = -\sin x$.

Pf

$$\sin y - \sin x = 2 \sin h \cos \frac{y+x}{2} \quad \left[h = \frac{y-x}{2}\right]$$

$$= 2 \sin h \cos(x+h)$$

So
$$\lim_{y \to x} \frac{\sin y - \sin x}{y-x} = \lim_{h \to 0} \frac{2 \sin h}{2h} \cos(x+h)$$

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$$= \lim_{h\to 0} \frac{\sin h}{h} \times \lim_{h\to 0} \cos(x+h)$$

AoL (allowable since both of these limits exist)

$$= \underbrace{1}_{\text{by PROP}} \times \underbrace{\cos(x+0)}_{\text{by continuity of cos}} = \cos x.$$

So $\sin' x = \cos x$ (by Def^n).

Derivative of $\cos x$: similar (left to the student). $\blacksquare$