MATH11112 REAL ANALYSIS Week Ø5 lecture A

Check In

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Reminder

- We proved the Product Rule* and the Chain Rule of differentiation

* Suppose f and g are diff^{ble} at a, then fg is diff^{ble} at a and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$

- Need to deduce the Quotient Rule (week 4 notes) and the Inverse Rule ← differentiate $\ln$

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Corollary (the Quotient Rule)

Assume $g(a) \neq 0$ and f, g are differentiable at a.

Then $\frac{1}{g}$, $\frac{f}{g}$ are diff'ble at a and

$$\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{g(a)^2} \qquad \left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

Proof

$$\frac{d}{dx}\left(\frac{1}{x}\right) \underset{x\neq 0}{=} \lim_{y \to \cancel{x}} \frac{\frac{1}{y} - \frac{1}{x}}{y - x} = \lim_{\substack{\cancel{x\neq 0} \\ y \neq 0}} \underbrace{\lim_{y \to x}}_{} \frac{yx\left(\frac{1}{y} - \frac{1}{x}\right)}{yx(y-x)} =$$

in some open nbhd of x, there is no 0 so we can assume $y \neq 0$

$$= \lim_{y \to x} \frac{\cancel{x - y}\,(-1)}{yx\cancel{(y-x)}}$$

$$= \lim_{y \to x} \frac{-1}{yx} \underset{\text{AoL}}{=} -\frac{1}{x^2}.$$

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$h(x) = \frac{1}{x}$ $\qquad \frac{1}{g(y)} = h(g(y))$, so to differentiate this function, we use the chain rule:

$$\left.\begin{array}{l} g \text{ is diff}^{\text{ble}} \text{ at } a \\ g(a) \neq 0 \Rightarrow h \text{ is diff}^{\text{ble}} \text{ at } g(a) \end{array}\right]$$ assumptions in the Chain Rule checked

By the Chain Rule, $(h \circ g)'(a) =$

$$= h'(g(a))\, g'(a) = -\frac{1}{g(a)^2}\, g'(a) = \frac{-g'(a)}{g(a)^2}.$$

To deduce $\left(\frac{f}{g}\right)'(a)$, write $\frac{f}{g} = f \times \frac{1}{g}$ and apply the Product Rule.

THM (The Inverse Rule) Let $f(x)$ be strictly monotonic and continuous on $[a,b]$.
Let g be the inverse of f (so that g is strictly monotonic and continuous by the Inverse Function Thm).
Suppose f is diff^{ble} at $\ell \in (a,b)$ and $f'(\ell) \neq 0$.

Then g is diff^{able} at $f(\ell)$ and

$$g'(f(\ell)) = \frac{1}{f'(\ell)}.$$

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WRONG "proof"

g inverse to $f \Rightarrow g \circ f = \text{identity}$

so $g(f(x)) = x \quad \forall x \in [a,b]$.

Apply the Chain Rule:

$g'(f(\ell)) \cdot f'(\ell) = x' = 1$

so $g'(f(\ell)) = \frac{1}{f'(\ell)}$.

WRONG!!! I applied the Chain Rule without checking the assumptions made in the Chain Rule Theorem.

ASSUMPTION: g is diff^ble at $f(\ell)$

This doesn't prove that g is diff'able at $f(\ell)$.

Correct proof. By an earlier Proposition, "f is diff^{ble} at ℓ" means that we can find a continuous at ℓ function $F_\ell(x)$ ("slope function at ℓ") such that

$$f(x) - f(\ell) = F_\ell(x)(x-\ell) \quad (\forall x)$$

This in particular holds for $x = g(y)$ (also, put $k = f(\ell)$ so that $g(k) = \ell$)

$$f(g(y)) - f(g(k)) = F_\ell(g(y)) \cdot \big(g(y) - g(k)\big)$$

for all y where $g(y)$ is defined

$[f(g(y)) = y \quad \forall y]$

so
$$y - k = F_\ell(g(y)) \cdot (g(y) - g(k))$$

for all y where $g(y)$ is defined

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If $y \neq k$, then $y-k \neq 0 \Rightarrow F_\ell(g(y))(g(y)-g(k)) \neq 0$
$\Rightarrow F_\ell(g(y)) \neq 0$.

If $y=k$:
$F_\ell(g(k)) = F_\ell(\ell) = f'(\ell) \underbrace{\neq 0}_{\text{by assumption}}$

So, $F_\ell(g(y))$ is never 0, so we can divide by it:

$$g(y)-g(k) = \frac{1}{F_\ell(g(y))}(y-k) \quad \forall y.$$

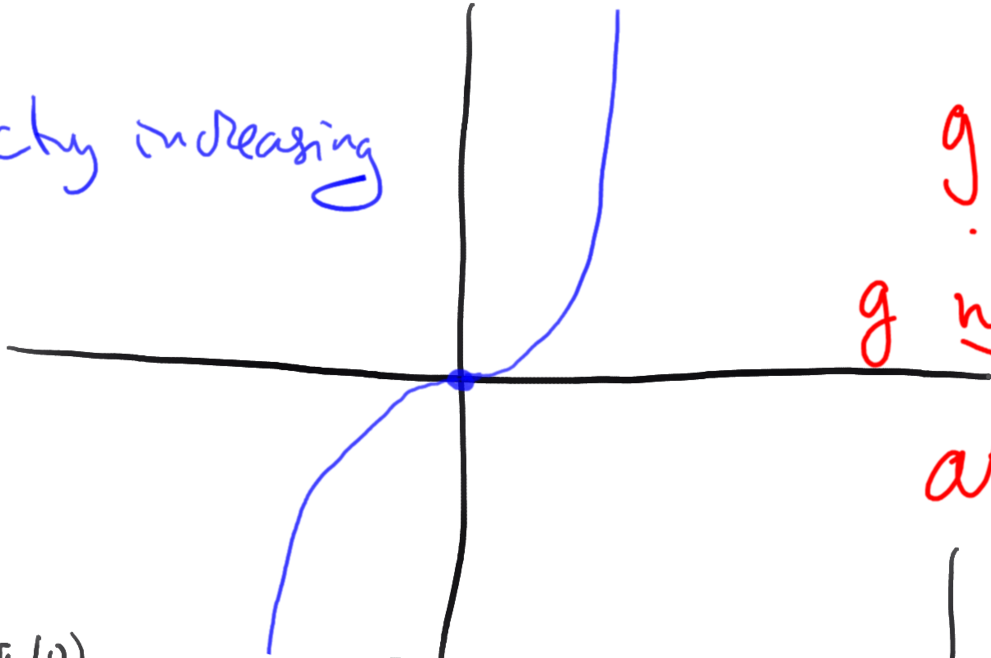
continuous at k because g is continuous at k and F_ℓ is ct^s at $\ell = g(k)$
composition & algebra of ct^s fns

By the same Prop'n again, g is diff^{ble} at $y=k$ and $g'(k) = \frac{1}{F_\ell(g(k))}$
$= 1/F_\ell(\ell) = 1/f'(\ell)$

EX f can be strictly increasing and diff'ble but at a point c one can still have $f'(c)=0$.

$f(x)=x^3$ strictly increasing

$f'(0)=0$

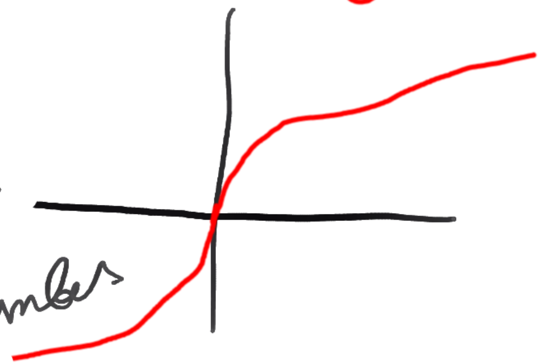


$g(y)=\sqrt[3]{y}$

g not diff'ble at $y=0$:

$$\lim_{y\to 0}\frac{g(y)-g(0)}{y-0}=\lim_{y\to 0}\frac{\sqrt[3]{y}}{y}=\lim_{y\to 0}\frac{1}{y^{2/3}}$$

not a real number



Corollary $\frac{d}{dy} \ln y = \frac{1}{y}$ $(\forall y > 0)$

Proof $\ln = \exp^{-1}$ so by the inverse rule

$$\frac{d}{dy} \ln(y) = \frac{1}{\exp'(x)} = \frac{1}{\underset{\neq 0}{\exp(x)}}$$

$(y = e^x)$

$(x = \ln y)$

$$= \frac{1}{\exp(\ln y)} = \frac{1}{y} \quad \blacksquare$$

exercise

$b \in \mathbb{R}$, $x^b = e^{b \ln(x)}$

$(x > 0)$ Show: $(x^b)' = b x^{b-1}$