MATH11112 Real Analysis (2025). Exercise sheet for week 03

Convergence tests

Check In

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Week 04 : Examples Class

Note :

- there will be no examples class in Week 06,
- because you will be doing the Online Coursework Test (worth 10% of the final mark for MATH11112) (timed: 30 minutes) Submission window: 24h

Supervision work

Attempt these questions and bring your solutions to the Week 3 supervision class for discussion.

Q10. Exercise on the Ratio Test. The **Fibonacci sequence** $F_1, F_2, F_3, \ldots$ is defined by the rule $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_{n-1} + F_n$ for all $n \geq 2$. Consider the **Fibonacci reciprocal series**

$$\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{F_n}.$$

(i) Write down the first six terms of the series.

(ii) Write down the explicit formula for F_n in terms of the two roots, α and β, of the quadratic equation $x^2 = x + 1$. This formula should be known to you from Section 2 of MFA (Semester 1).

(iii) Apply the Ratio Test to the series $\sum_{n=1}^{\infty} \frac{1}{F_n}$. Use the formula from (ii) to find the limit ℓ needed for the test. Then, based on the numerical value of ℓ, determine whether the series is convergent.

Q11. Exercises on the Comparison Test. (i) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent, by comparing it with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(ii) Show that $\sum_{n=1}^{\infty} \frac{100}{n^2 - 0.5\sqrt{n}}$ is convergent, by comparing it with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, multiplied by a suitable constant.

(iii) Are $\sum_{n=1}^{\infty} \frac{1}{n-0.5}$ and $\sum_{n=1}^{\infty} \frac{1}{n+100.5}$ convergent or divergent?

Q12. (a question from a past exam paper) Use Algebra of Infinite Sums to prove that the following series is convergent: $\sum_{n=1}^{\infty} \left(\frac{1000}{n^2} + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$.

Q13. Absolute vs. conditional convergence. (i) Show: $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \ldots$ is a convergent series with sum 0.

(ii) Determine whether the series in (i) is **absolutely convergent** or **conditionally convergent.** (By definition, "conditionally convergent" means convergent but not absolutely convergent.)

Extra exercises

Attempt these questions in your own time and compare your answers with the model solutions published on Monday in week 4. Some of these questions may be discussed in the Examples Class on Thursday in week 4.

* **Q14. (A question used in a past exam paper.)** Determine whether the following series converge. In each case you should briefly justify your answer (in particular, saying what test you are using).

$$(a)\ \sum_{n=1}^{\infty} \frac{5^n}{2^n \cdot n^3} \qquad (b)\ \sum_{n=2}^{\infty} \frac{4n^3}{n^4 - 1} \qquad (c)\ \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}.$$

(a) Since we have terms of the form c^n, it is advisable to try the Ratio Test. 1. The terms are positive: ✓

2. $$\ell = \lim_{n\to\infty} \frac{\frac{5^{n+1}}{2^{n+1}(n+1)^3}}{\frac{5^n}{2^n n^3}} = \lim_{n\to\infty} \left(\frac{5}{2} \left(\frac{n}{n+1}\right)^3 \right)$$

$$= \lim_{n\to\infty} \frac{5}{2} \left(\frac{1}{1+\frac{1}{n}}\right)^3 = \cancel{\lim_{n\to\infty}}\ \frac{5}{2}\left(\frac{1}{1+0}\right) = \frac{5}{2} > 1$$

Since $\ell > 1$, by the Ratio Test, the series is divergent.

(b) $$\frac{4n^3}{n^4-1} = \frac{4}{\underbrace{n - \frac{1}{n^3}}_{<n}} > \frac{1}{n}$$ $(n \geq 2)$

We compare this series to the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$

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Partial sums are unbounded (they are partial sums of the harmonic series, minus 1)
$\Rightarrow$ the series is divergent.

(c) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \ldots\ldots$

$= (-1) \times$ alternating harmonic series.

Convergent, by the Alternating Series Test.

Q15. (A question used in a past exam paper.) Find ~~an $R \in [0,\infty)$ such that the series~~ the radius of convergence $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ ~~is absolutely convergent when $|x| < R$ and is divergent when $|x| > R$.~~ of the power series.

Solution: need to find

$$\ell = \lim_{n\to\infty} \frac{((n+1)!)^2 \, |x|^{n+1} \, (2n)!}{(2n+2)! \, (n!)^2 \, |x|^n} = \lim_{n\to\infty} \frac{|x|(n+1)^2}{(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} |x| \frac{(1+1/n)^2}{(2+1/n)(2+2/n)}$$

$$= |x| \frac{(1+0)^2}{(2+0)(2+0)} = \frac{|x|}{4}$$

By Ratio Test, the series is absolutely convergent when $|x| < 4$ and is not absolutely convergent when $|x| > 4$

CONCLUSION: $R = 4$.

Q16. Construct non-trivial examples of convergence. Give an example of:

(i) a convergent series $\sum_{n=1}^{\infty} a_n$ such that the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is divergent;

Observation: $|(-1)^n a_n| = |a_n|$ so if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then so is $\sum_{n=1}^{\infty} (-1)^n a_n$.

Put $a_n = \frac{(-1)^n}{n}$ (Alternating Harmonic Series)

$(-1)^n a_n = \frac{1}{n}$ $\qquad \sum \frac{1}{n} = +\infty$

(ii) a convergent series $\sum_{n=1}^{\infty} a_n$ and a sequence $\lambda_n \to 0$ such that the series $\sum_{n=1}^{\infty} \lambda_n a_n$ is divergent;

(iii) a convergent series $\sum_{n=1}^{\infty} a_n$ and nonnegative $\lambda_n \to 0$ such that $\sum_{n=1}^{\infty} \lambda_n a_n$ is divergent.

$a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$ (conditionally convergent by the Alt Series Test)

$\lambda_n = \begin{cases} 0, & n \text{ even} \\ \frac{1}{\sqrt{n}}, & n \text{ odd} \end{cases}$ $\qquad \sum \lambda_n a_n = \frac{1}{\sqrt{1}} + 0 + \frac{1}{\sqrt{3}} + 0 + \frac{1}{\sqrt{5}} + 0 \ldots$

if n is odd, $\qquad$ nth partial sum $\geq n/2 \left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2}\sqrt{n}$

Partial sums are unbounded. $\rightarrow +\infty$

Q17. Rearrangements. As shown in Q13, the series $1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\ldots$ is convergent with sum 0. Now consider its rearrangement

$$1+\frac{1}{2}-1+\frac{1}{3}+\frac{1}{4}-\frac{1}{2}+\frac{1}{5}+\frac{1}{6}-\frac{1}{3}+\frac{1}{7}+\frac{1}{8}-\frac{1}{4}+\ldots$$

where the pattern is, two positive terms followed by one negative term. Show that this rearrangement **does not** have sum 0. (Hint: consider the partial sum s_{3n} and let $n\to\infty$.)

Why does this example not contradict the rearrangement theorems proved in lectures?

$$1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\ldots$$

$1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots$

Partial sums:

$0 \le S_n \le \frac{1}{n/2}$

$0 \to 0$, $\frac{1}{n/2} \to 0$ as $n\to\infty$

By Sandwich, $S_n \to 0$.

The series is convergent with sum 0.

$$\underbrace{1+\frac{1}{2}}_{S_3=\frac{1}{2}}\underbrace{-1+\frac{1}{3}+\frac{1}{4}}\underbrace{-\frac{1}{2}+\frac{1}{5}+\frac{1}{6}}\underbrace{-\frac{1}{3}+\frac{1}{7}+\frac{1}{8}}-\frac{1}{4}+\ldots$$

$> \frac{1}{4}+\frac{1}{4}-\frac{1}{2} = 0$ $\quad > \frac{1}{6}+\frac{1}{6}-\frac{1}{3} = 0$

$S_6 > S_3 \quad S_9 > S_6 \quad \ldots\ldots \quad$ etc

So $S_{3n} > S_3 = \frac{1}{2}$ and $\lim_{n\to\infty} S_n \neq 0$