MATH11112 REAL ANALYSIS **Week** 04 **lecture** B

Check In

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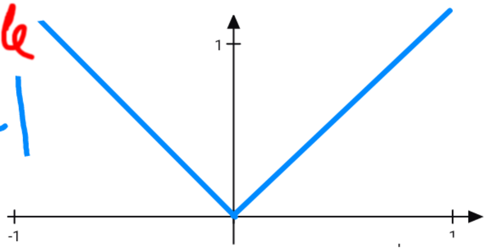
At the end of the last lecture, we proved:

f is diff'able at a $\Longrightarrow$ f is continuous at a.

NOTE THAT "$\Leftarrow$" IS, IN GENERAL, FALSE:

counterexample $f(x) = |x|$



f is continuous at all points including $x=0$

$$\lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-} \frac{|x|}{x} = \lim_{x\to 0-} \frac{-x}{x} = -1$$

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{|x|}{x} = \lim_{x\to 0+} \frac{x}{x} = 1$$

$\lim_{x\to 0-} \neq \lim_{x\to 0+}$ so $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ DOES NOT EXIST:

f is NOT diff'able at $x=0$.

We now know how to differentiate $\begin{cases} \text{constants} \\ f(x) = x \end{cases}$.
Not enough?

THM (sum & product rules) Suppose f, g are diff'ble at a.
Then
- exercise → $*$ $f+g$ is diff'ble at a, $(f+g)'(a) = f'(a) + g'(a)$
- $*$ fg is diff'ble at a, $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Pf

$$\frac{f(x)g(x) - f(a)g(a)}{x-a} = \frac{f(x)g(x) - f(a)g(x)}{x-a} + \frac{f(a)g(x) - f(a)g(a)}{x-a}$$

$$= \frac{f(x) - f(a)}{x-a}\, g(x) + f(a)\, \frac{g(x) - g(a)}{x-a}$$

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$$\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}=\lim_{x\to a}\left(\frac{f(x)g(x)-f(a)g(x)}{x-a}+\frac{f(a)g(x)-f(a)g(a)}{x-a}\right)$$

$$=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}g(x)+\lim_{x\to a}f(a)\frac{g(x)-g(a)}{x-a}$$

both limits exist (see below), so we can use AoL

$$\lim_{x\to a}\underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow{x\to a} f'(a)}\underbrace{g(x)}_{\to g(a)}=f'(a)g(a)$$

g is continuous at a (diff'able $\Rightarrow$ ct^s)

by defⁿ of f' | by criterion of continuity

$$\lim_{x\to a}\underbrace{f(a)}_{\text{constant}}\underbrace{\frac{g(x)-g(a)}{x-a}}_{\to g'(a)}$$

$$=f(a)g'(a).$$

$\blacksquare$

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<u>Note</u> All polynomials in x are diff'ble.

How to differentiate e^x ?

NOT JUSTIFIED: $\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)' = \sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)'$

$\frac{x^k}{k!} = \frac{kx^{k-1}}{k!} = \frac{x^{k-1}}{(k-1)!}$

* could justify for <u>power series</u>

So $(e^x)' = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$

<u>PROP</u> A function $f(x)$, defined in an open nbhd of a, is diff'ble at a, if and only if there exists a function $F_a(x)$ with $f(x) - f(a) = F_a(x)(x-a)$ for all x, and $F_a(x)$ is continuous at $x=a$. [slope function]

If this holds, $f'(a) = F_a(a)$

Pf If $F_a(x)$ exists and is cts at a, we have

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \overset{\uparrow}{\underset{x\neq a}{=}} \lim_{x\to a} F_a(x) \underset{\text{criterion of continuity}}{=} F_a(a).$$

If f is diff'able at a, define:

$$F_a(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a}, & \text{if } x\neq a \\ f'(a), & \text{if } x=a. \end{cases}$$

Then $\displaystyle\lim_{x\to a} F_a(x) \overset{\uparrow}{\underset{x\neq a}{=}} \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \underset{\text{def}}{=} f'(a) = F_a(a)$

which proves that F_a is cts at a.

$\blacksquare$

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~~TH~~ PROP $\quad \frac{d}{dx} e^x = e^x$.

Proof First, differentiate e^x at 0:

$$e^x - e^0 = (x-0)\,F_0(x) = \underline{x\,F_0(x)} \qquad F_0 = ?$$

$$\parallel$$

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \underset{\text{AoIS}}{=} x \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!}$$

take x out of each term

So $F_0(x) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!}$ is given by a power series which has infinite radius of convergence; therefore, $F_0(x)$ is continuous for all x. Hence: the derivative of e^x at 0 is $F_0(0) = 1$.

PROVE SPECIAL LIMIT for e^x: $\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1}$

Differentiate at any x:

$$e^y = e^x e^{y-x}$$

$$(e^x)' = \lim_{y\to x} \frac{e^y - e^x}{y-x} = \lim_{y\to x} e^x \frac{e^{y-x}-1}{y-x}$$

$$= e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x . \blacksquare$$

$h = y - x$

Can I now differentiate $\dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$?

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

THM (the Chain Rule)

If $g(y)$ is diff'ble at $y = k$
and $f(x)$ is diff'ble at $x = g(k)$
then $(f\circ g)(y)$ is diff'ble at $y = k$ and

$$(f\circ g)'(k) = f'(g(k))\, g'(k).$$

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Pf

$$g(y) - g(k) = \underbrace{G_k(y)}_{\text{ct}^s \text{ at } y=k} (y-k)$$

if f is diff'ble at ℓ

$$f(x) - f(\ell) = \underbrace{F_\ell(x)}_{\text{ct}^s \text{ at } x=\ell} (x-\ell)$$

This equation holds for all x, in particular for $x = g(y)$:

Put $\ell = g(k)$.

$$f(g(y)) - f(g(k)) = F_\ell(g(y))\,(g(y) - g(k))$$

$$= \underbrace{F_\ell(g(y))\,G_k(y)}\,(y-k)$$

by continuity of composition, $F_\ell(g(y))$ is ct^s at k

by A. Cont Fns, this is ct^s at k. So $f(g(y))$ is diff'ble and its derivative at k is $F_\ell(\ell)G_k(k) = f'(\ell)\,g'(k)$. $\blacksquare$