MATH11112 REAL ANALYSIS **Week** 04 **lecture** A

SEAtS 042401

Check In ?

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Today, we will study the function

$$\exp(x) = e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

introduce $\ln(x)$

and differentiation.

Def $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ (note: convergent for all x)

$\exp$ is a function defined on $\mathbb{R}$.

THM $\exp(x)$ is continuous on $\mathbb{R}$. $\forall x, y \in \mathbb{R}$
$\exp(x)\exp(y) = \exp(x+y)$

Pf Continuity is by result from last week:
sum of a power series is continuous for $x \in (-R, R)$ where $R =$ radius of convergence.
for exp, $R = \infty \Rightarrow$ continuity for all $x \in \mathbb{R}$.

Consider the double series $a_{m,n} = \frac{x^m}{m!} \frac{y^n}{n!}$.

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$$\begin{array}{ccccc}
1 & y & \frac{y^2}{2!} & \frac{y^3}{3!} & \cdots \\
x & xy & x\,y^2/2! & x\,\frac{y^3}{3!} & \cdots \\
\frac{x^2}{2!} & \frac{x^2}{2!}y & \frac{x^2}{2!}\,\frac{y^2}{2!} & \frac{x^2}{2!}\,\frac{y^3}{3!} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

$$\mathrm{RowSum}_m = \frac{x^m}{m!}\sum_{n=0}^{\infty}\frac{y^n}{n!} = \frac{x^m}{m!}\exp(y)$$

$$\mathrm{DiagSum}_d = \frac{1}{d!}\sum_{k=0}^{d} d!\,\frac{x^{d-k}}{(d-k)!}\,\frac{y^k}{k!}$$

$$\sum_{m=0}^{\infty}\mathrm{RowSum}_m \overset{\text{AoIS}}{=} \sum_{m=0}^{\infty}\frac{x^m}{m!}\exp(y) = \exp(x)\exp(y)$$

observe : $\dfrac{d!}{(d-k)!\,k!}$ is the binomial coefficient, $\binom{d}{k}$

so $\mathrm{DiagSum}_d = \dfrac{1}{d!}(x+y)^d$ (binomial theorem)

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So $\displaystyle\sum_{d=0}^{\infty} \text{DiagSum}_d = \sum_{d=0}^{\infty} \frac{1}{d!}(x+y)^d$

$= \exp(x+y)$.

① If $x, y \geq 0$: sum of entries in ~~the~~ the double series does not depend on the method of summation, so

$\exp(x)\exp(y) = \exp(x+y)$

② General case:

$$\sum_{m,n} \left| \frac{x^m}{m!} \frac{y^n}{n!} \right| = \sum_{m,n} \frac{|x|^m}{m!} \frac{|y|^n}{n!} = \exp(|x|) \times \exp(|y|) < \infty$$

so, again $\exp(x)\exp(y) = \exp(x+y)$ $\square$

This motivates:

$n \in \mathbb{N}$ $\quad \exp(n) = \exp(\overbrace{1+1+\dots+1}^{n}) = \exp(1)^n = e^n$

NOTATION: $\exp(1) = e$

$\exp(0) = 1$; $\exp(-n) \underset{\uparrow}{=} \dfrac{1}{\exp(n)} = \dfrac{1}{e^n} = e^{-n}$

THM

FOR ALL $x \in \mathbb{R}$, $\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Prop The function $f(x) = e^x$ is a strictly increasing bijection $\mathbb{R} \to (0, +\infty)$.

$x \mapsto e^x$

Pf If $x > 0$: $e^x = 1 + x + \frac{x^2}{2!} + \dots > 1 + x > 0$

also $e^{-x} = \frac{1}{e^x} > 0$, so $e^x \in (0, +\infty)$ $\forall x$.

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Strictly Increasing: $x<y$ $\qquad e^y - e^x = e^x(\underbrace{e^{y-x} - 1}_{> 1 + (y-x) - 1 > 0})$

and so $e^y > e^x$

Strictly Increasing $\Rightarrow$ injective

Surjective: let $d \in (0, +\infty)$.

If $d > 1$: $\quad e^d > 1 + d > d$ *

$e^{-d} = \frac{1}{e^d} < \frac{1}{d} < 1 < d$ *

d

IVT $\Rightarrow$ $\exists c \in \mathbb{R}$: $e^c = d$.

e^x continuous

If $d < 1$: $\exists c$: $e^c = \frac{1}{d} > 1$, so $e^{-c} = d$ $\blacksquare$

If $d = 1$: $\quad e^0 = 1$

Define $\ln : (0,+\infty) \longrightarrow \mathbb{R}$

$$\ln = (\exp)^{-1}$$

THM The function $\ln : (0,+\infty) \to \mathbb{R}$ is strictly increasing bijection, continuous,

$$\ln(e^x) = x \quad (\forall x \in \mathbb{R}),$$

$$e^{\ln(y)} = y \quad (\forall y \in (0,+\infty))$$

$$\ln(yz) = \ln(y) + \ln(z) \quad \forall y, z \in (0,+\infty)$$

$\blacksquare$

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Def Open neighbourhood of a $(a \in \mathbb{R})$:
interval $(a-\delta, a+\delta)$, $\delta > 0$

Let f be a function defined in an open nbhd of $a \in \mathbb{R}$. We say that f is differentiable at a if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. The (numerical) value of this limit is the derivative, $f'(a)$, of f at a.

Def If f is diff'able at all points of an open interval, we have the function $f'(x)$ defined in that interval: the derivative of f

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Ex Constant f^n: $f(x) = c \quad \forall x \in \mathbb{R}$

is diff'able: $a \in \mathbb{R}$, ..

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \underbrace{\frac{c - c}{x - a}}_{\text{undefined if } x = a} \circledast$$

but $\lim_{x \to a}$ does not care about the value at $x = a$, or about whether this value exists.

$=$ can assume $x \neq a$

$$\lim_{x \to a} \frac{0}{x - a} = \lim_{x \to a} 0 = 0.$$

$$\boxed{\frac{d}{dx}(\text{const}) = 0}$$

Ex $f(x)=x$: $\lim_{x\to a} \frac{x-a}{x-a} = 1$ $\left(\frac{d}{dx}(x) = 1 \right)$

THM If f is diff'able at a, then f is continuous at a.

Pf Continuous at $a \iff \lim_{x\to a} f(x) = f(a)$

$$\underset{\text{AoL}}{\iff} \lim_{x\to a} \left(f(x)-f(a)\right) = 0 \quad (*)$$

Suppose f is diff'able at a: $L = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.

$$\lim_{x\to a} \left(f(x)-f(a)\right) \overset{\text{AoL}}{=} \underbrace{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}\ \underbrace{\lim_{x\to a} (x-a)}$$

Both limits exist!!!

$= L \cdot 0 = 0.$ $(*)$ proved ∎